Section 2 – Solutions

Review

1) **Subsets/Binomial Coefficients** The number of ways to choose a k-element subset of a set of n elements is _____.

2) **Binomial theorem.** $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x+y)^n =$ ____________________.

3) **Inclusion-exclusion.** $|A \cup B| =$ ____________________.

4) **Inclusion-exclusion.** $|A \cup B \cup C| =$ ________________________________.

5) **Pigeonhole principle.** If there are n pigeons and k holes, and $n > k$, some hole has at least ____________________ pigeons.

6) **Multinomial coefficients.** Suppose there are n objects, but only k are distinct, with $k \leqslant n$. (For example, "godoggy" has $n = 7$ objects (characters) but only $k = 4$ are distinct: (g, o, d, y)). Let n_i be the number of times object i appears, for $i \in \{1, 2, \ldots, k\}$. (For example, $(3, 2, 1, 1)$, continuing the "godoggy" example.) The number of distinct ways to arrange the n objects is: ________________________________

7) **Binary encoding(a.k.a Stars and Bars).** The number of ways to distribute n indistinguishable balls into k distinguishable bins is ________________________________.

8) **Probability space.** We call the set of possible outcomes of an experiment the **sample space**(denoted Ω), $\omega \in \Omega$ an **outcome** of the experiment, $E \subseteq \Omega$ an **event**, and $\mathbb{P} : \Omega \to [0, 1]$ the **probability measure/function**. A **probability space** is a pair $(\Omega, \mathbb{P})$ where we have $\mathbb{P}(\omega)$ ______ for all $\omega \in \Omega$ and $\sum_{\omega \in \Omega} \mathbb{P}(\omega) =$ ______.

9) **Mutually exclusive events.** The events $\mathcal{A}$ and $\mathcal{B}$ are *mutually exclusive* if $\mathcal{A} \cap \mathcal{B} =$ __________

10) **Additivity of Probability.** If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are mutually exclusive events, then $\mathbb{P}\left(\bigcup_{i=1}^n \mathcal{A}_i\right) =$ _______________ .

11) **Complement.** For any event $\mathcal{A}$, $\mathbb{P}(\mathcal{A}^c) =$ _______________ .

12) **Uniform Probability Space.** A probability space $(\Omega, \mathbb{P})$ where $\mathbb{P}(\omega) =$ _______________ for all $\omega \in \Omega$

13) **Equally Likely Outcomes.** If every outcome in a finite sample space Ω is equally likely(i.e. we have an uniform prob. space), and E is an event, then $\mathbb{P}(E) =$ _______________ .

Task 1 – HBCDEFGA

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

> The total number of permutations is $8!$. The number of permutations with A at the beginning is $7!$ and the number with H at the end is $7!$. By inclusion/exclusion, the number that have either A at the beginning or H at the end or both is $2 \cdot 7! - 6!$ since there are $6!$ that have A at the beginning and H at the end. Finally, using complementary counting, the number that have neither A at the end or H at the end is $8! - (2 \cdot 7! - 6!)$.

Task 2 – Ingredients

Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent.

> We use inclusion-exclusion. Let A be the set of all anagrams (permutations) of "INGREDIENT", and A_I be the set of all anagrams with two consecutive I's. Define A_E and A_N similarly. $A_I \cup A_E \cup A_N$ clearly are the set of anagrams we don't want. So we use complementing to count the size of $A \backslash (A_I \cup A_E \cup A_N)$. By inclusion exclusion, $|A_I \cup A_E \cup A_N|$ =singles-doubles+triples, and since $A_I \cup A_E \cup A_N \subseteq A$, by complementing, $|A \backslash (A_I \cup A_E \cup A_N)| = |A| - |A_I \cup A_E \cup A_N|$.
>
> First, $|A| = \frac{10!}{2!2!2!}$ because there are 2 of each of I,E,N's (multinomial coefficient). Clearly, the size of A_I is the same as A_E and A_N. So $|A_I| = \frac{9!}{2!2!}$ because we treat the two adjacent I's as one entity. We also need $|A_I \cap A_E| = \frac{8!}{2!}$ because we treat the two adjacent I's as one entity and the two adjacent E's as one entity (same for all doubles). Finally, $|A_I \cap A_E \cap A_N| = 7!$ since we treat each pair of adjacent I's, E's, and N's as one entity.
>
> Putting this together gives $\boxed{\frac{10!}{2!2!2!} - \left(\binom{3}{1} \cdot \frac{9!}{2!2!} - \binom{3}{2} \cdot \frac{8!}{2!} + \binom{3}{3} \cdot 7!\right)}$

Task 3 – Card Party

At a card party, someone brings out a deck of bridge cards (4 suits with 13 cards in each). N people each pick 2 cards from the deck and hold onto them. What is the minimum value of N that guarantees at least 2 people have the same combination of suits?

> $\boxed{N = 11}$: There are $\binom{4}{2}$ combinations of 2 different suits, plus 4 possibilities of having 2 cards of the same suit. With $N = 11$ you can apply the pigeonhole principle.

Task 4 – The Pigeonhole Principle

Show that in any group of n people there are two who have an identical number of friends within the group. (Friendship is bi-directional – i.e., if A is friend of B, then B is friend of A – and nobody is a friend of themselves.)

Solve in particular the following two cases individually:

a) Everyone has at least one friend.

> Everyone has between 1 and $n-1$ friends (i.e., $n-1$ holes), and there are n people (the "pigeons"). Therefore, two of them will have the same number of friends.

b) At least one person has no friends.

> Here, we need to observe that if someone has 0 friends, then nobody has $n-1$ friends (by the symmetry of the friendship relation). Then, possible choices are now between 0 and $n-2$ friends (i.e., $n-1$ holes), and there are n people (the "pigeons"). Therefore, two of them will have the same number of friends.
>
> Alternatively, we can split into cases.
> Case 1 (at least 2 people have no friends): In this case, there's two people that have the same number of friends (0).

Case 2 (exactly 1 person has no friends): If exactly 1 person has 0 friends, then that means there's $n-1$ people with at least one friend. In this group, everyone has between 1 and $n-2$ friends (since no one is friends with the one person with no friends) (i.e., $n-2$ pigeonholes), and since there's $n-1$ people/pigeons, at least two must have the same number of friends.

Task 5 – A Team and a Captain

Give a combinatorial proof of the following identity:

$$n\binom{n-1}{r-1} = \binom{n}{r}r.$$

Hint: Consider two ways to choose a team of size r out of a set of size n and a captain of the team (who is also one of the team members).

Remember that a combinatorial proof just requires that we show both sides are equivalent ways of counting a situation.

Left hand side: Choose a team of size r and a captain for that team (from among the r) by first choosing the captain (n choices) and then choosing the rest of the team $\binom{n-1}{r-1}$.

Right hand side: Choose a team of size r and a captain for that team by first choosing the team ($\binom{n}{r}$ choices) and then choosing the captain from among the members of the team (r choices).

Task 6 – Balls from an Urn

Suppose that an urn (a fancy name for a jar that doesn't have a lid) contains one red ball, one blue ball, and one green ball. (Other than their colors, balls are identical.) Imagine we draw two balls *with replacement*, i.e., after drawing one ball we put it back into the urn before we draw the second one. (In particular, each ball is equally likely to be drawn.)

a) Give a probability space describing the experiment.

$\Omega = \{B, R, G\}^2$ and $\mathbb{P}(\omega) = 1/9$ for all $\omega \in \Omega$.

b) What is the probability that both balls are red? (Describe the event first, before you compute its probability.)

The event is $\mathcal{A} = \{RR\}$. Its probability is $\mathbb{P}(\mathcal{A}) = \frac{|\mathcal{A}|}{9} = \frac{1}{9}$.

c) What is the probability that at most one ball is red?

This is just $\mathcal{A}^c$, the complement of $\mathcal{A}$. We know that $\mathbb{P}(\mathcal{A}^c) = 1 - \mathbb{P}(\mathcal{A}) = 1 - \frac{1}{9} = 8/9$.

d) What is the probability that we get at least one green ball?

This is the event $\mathcal{B} = \{GR, GB, GG, RG, BG\}$, and thus $\mathbb{P}(\mathcal{B}) = \frac{|\mathcal{B}|}{9} = \frac{5}{9}$.

e) Repeat **c)-d)** for the case where the balls are drawn *without replacement*, i.e., when the first ball is drawn, it is not placed back from the urn.

Here, the probability space changes: First of all, the outcomes RR, GG, BB are not possible any more, so let us remove them from Ω, which is now $\Omega = \{BG, BR, GB, GR, RB, RG\}$. Note that now we have $\mathbb{P}(\omega) = 1/3 \cdot 1/2 = 1/6$ for every outcome, because we have three choices for the first ball, but only two for the second.
It can never be that both balls are red – therefore, for **c)**, the probability becomes 1 (i.e., the associated event *is* Ω.) For **d)**, instead, the event becomes $\mathcal{B} = \{GR, GB, RG, BG\}$, and $\mathbb{P}(\mathcal{B}) = 4 \cdot \frac{1}{6} = \frac{2}{3}$.

Task 7 – Shuffling Cards

We have a deck of cards, with 4 suits, and 13 cards in each suit. Within each suit, the cards are ordered Ace > King > Queen > Jack > 10 > $\cdots$ > 2. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely).

What is the probability the first card on the deck is (strictly) larger than the second one?

First off, the sample space Ω here consists of all pairs of cards – which we can represent by their value *and* suit, e.g., $(4\clubsuit, A\diamondsuit)$. There $52 \cdot 51 = 2652$ possible outcomes, therefore $\mathbb{P}(\omega) = \frac{1}{2652}$ for all $\omega \in \Omega$.

Let us now look at the size of the event $\mathcal{E}$ containing all pairs where the first card is strictly larger than the second. Then, the number of pairs of values of cards a and b where $a < b$ is exactly $\binom{13}{2} = 13 \cdot 6 = 78$. We can then assign suits to each of them – given the cards are different, all suits are possible for each, so there are $4^2 = 16$ choices. Thus, overall,

$$|\mathcal{E}| = 16 \cdot 78 = 1248 \, .$$

Therefore,

$$\mathbb{P}(\mathcal{E}) = \frac{|\mathcal{E}|}{|\Omega|} = \frac{16 \cdot 78}{52 \cdot 51} = \frac{13 \cdot 3 \cdot 2^5}{13 \cdot 3 \cdot 2^2 \cdot 17} = \frac{8}{17} \approx 0.47 \, .$$

Task 8 – Robot Wears Socks

Suppose Joe is a k-legged robot, who wears a sock and a shoe on each leg. Suppose he puts on k socks and k shoes in some order, each equally likely. Each action is specified by saying whether he puts on a sock or a shoe, and saying which leg he puts it on. In how many ways can he put on his socks and shoes in a valid order? We say an ordering is valid if, for every leg, the sock gets put on before the shoe. Assume all socks are indistinguishable from each other, and all shoes are indistinguishable from each other.

First, note there are $2k$ objects — k shoes and k socks. Suppose we describe a sequence of actions, $Sock_1, Shoe_1, Sock_2, Shoe_2, \ldots, Sock_k, Shoe_k$.

This particular example means we first put a sock on leg 1, then a shoe on leg 1, then a sock on leg 2, etc. There are $(2k)!$ ways to order these actions. However, for each leg, there is only one valid ordering: the sock must come before the shoe. So we divide by 2^k and the total number of ways is $\frac{(2k)!}{2^k}$.

Alternatively, $\mathbb{P}(\text{valid ordering}) = \frac{|\text{valid orderings}|}{|\text{orderings}|}$, so $|\text{valid orderings}| = \mathbb{P}(\text{valid ordering}) * |\text{orderings}|$. We can compute $\mathbb{P}(\text{valid ordering}) = (1/2)^k$. Notice for any sequence of actions with each equally likely, the probability that the sock came before the shoe on a particular leg is $\frac{1}{2}$, so the probability this happened for each leg is $(1/2)^k$. Then $|\text{orderings}| = (2k)!$ because there are $2k$ actions that we can permute, all distinct. Multiplication gives the same answer as above.

Alternatively, we can think of there being $2k$ total, ordered actions, each action being either putting a sock on a certain leg or putting a shoe on a certain leg. We can first assign the pair of actions "put sock on leg 1" and "put shoe on leg 1" to two of the $2k$ actions. There are $\binom{2k}{2}$ ways to do this, as "put sock on leg 1" must come before "put shoe on leg 1". Next, we can assign "put sock on leg 2" and "put shoe on leg 2" to two of the $2k-2$ remaining actions. There are $\binom{2k-2}{2}$. We assign actions until we have 2 remaining actions for "put sock on leg k" and "put shoe on leg k". There are $\binom{2}{2} = 1$ ways to assign them. By the product rule, we have

$$\binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2} = \prod_{i=1}^{k}\binom{2i}{2}$$

total possible actions.

Task 9 – Congressional Tea

Twenty politicians are having tea, 6 Democrats and 14 Republicans.

a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans? (We assume every possible way of giving tea is equally likely.)

The sample space is the number of ways to give tea to people, so there are $\binom{20}{10}$ ways. The event is the ways to give tea to only Republicans, of which there are $\binom{14}{10}$ ways. So the probability is $\frac{\binom{14}{10}}{\binom{20}{10}}$.

b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 8 Republicans and 2 Democrats? (We assume every possible way of giving tea is equally likely.)

Similarly to the previous part, $\frac{\binom{14}{8}\binom{6}{2}}{\binom{20}{10}}$.

Task 10 – Trick or Treat

Suppose on Halloween, someone is too lazy to keep answering the door, and leaves a jar of exactly N total candies. You count that there are exactly K of them which are kit kats (and the rest are not). The sign says to please take exactly n candies. Each item is equally likely to be drawn. Let X be the number of kit kats we draw (out of n). What is $\mathbb{P}(X = k)$, that is, the probability we draw exactly k kit kats?

$$\boxed{\mathbb{P}(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}}$$

We choose k out of the K kit kats, and $n - k$ out of the $N - K$ other candies. The denominator is the total number of ways to choose n candies out of N total.

Task 11 – Binomial Theorem

What is the coefficient of z^{36} in $(-2x^2yz^3 + 5uv)^{312}$?

By the Binomial Theorem,

$$(-2x^2yz^3 + 5uv)^{312} = \sum_{k=0}^{312} \binom{312}{k} (-2x^2yz^3)^k (5uv)^{312-k} = \sum_{k=0}^{312} \binom{312}{k} (-2)^k x^{2k} y^k z^{3k} (5uv)^{312-k}$$

The term that gives z^{36} is the one with $k = 12$. Therefore, the coefficient is $\boxed{\binom{312}{12}(-2x^2y)^{12}(5uv)^{300}.}$