Quiz Section #7: Supplementary Exercises

CSE 312: Foundations of Computing II

Recall the probability density function for $X \sim \text{Exp}(\lambda)$:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , \text{ if } x \geq 0 \\ 0 & , \text{ if } x < 0 \end{cases}.$$

1. Starting from the probability density function, prove that $\text{E}[X] = 1/\lambda$. (Hint: use integration by parts.)

2. Starting from the probability density function, prove that $\text{P}(X \geq t) = e^{-\lambda t}$, for $t \geq 0$. As a corollary, show that the cumulative distribution function for X is $F(t) = 1 - e^{-\lambda t}$.

3. Prove the memorylessness property for the exponential distribution $\text{Exp}(\lambda)$: If s and t are nonnegative, then $\text{P}(X > s + t \mid X > s) = \text{P}(X > t)$.

4. Prove the memorylessness property for the geometric distribution $\text{geo}(p)$.

5. The *gamma distribution* $\text{Gamma}(r, \lambda)$ is defined to be the sum of r independent $\text{Exp}(\lambda)$ random variables. It represents the waiting time until the r-th event. If $X \sim \text{Gamma}(r, \lambda)$, find $\text{E}[X]$ and $\text{Var}(X)$. (Just as the exponential is the continuous analog of the geometric – both representing waiting time – the gamma is the continuous analog of the negative binomial – see Exercise #6 of practice midterm #1.)

6. Alex came up with a function that he thinks could represent a probability density function. He defined the potential pdf for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid pdf? If not, find a constant c such that the pdf $f(x) = \frac{c}{1+x^2}$ is valid. Then find $\text{E}[X]$. (Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $\tan \frac{\pi}{2} = \infty$, and $\tan 0 = 0$.)

7. Let $X \sim \text{Exp}(\lambda)$. For $t < \lambda$, find $M_X(t) = \text{E}[e^{tX}]$. M is called the *moment generating function* of X. Find $M'_X(0)$ and $M''_X(0)$. Do you notice any relationship between these two values and $\text{E}[X]$ and $\text{E}[X^2]$ (which are sometimes called the first and second *moments* of X)?

8. You throw a dart at an $s \times s$ square dartboard. The goal of this game is to get the dart to land as close to the lower left corner of the dartboard as possible. However, your aim is such that the dart is equally likely to land at any point on the dartboard. Let random variable X be the length of the side of the smallest square B in the lower left corner of the dartboard that contains the point where the dart lands. That is, the lower left corner of B must be the same point as the lower left corner of the dartboard, and the dart lands somewhere along the upper or right edge of B. For X, find the CDF, PDF, $\text{E}[X]$, and $\text{Var}(X)$.