CSE 312: Foundations of Computing II
Quiz Section #4: Discrete Random Variables

**Review/Mini-Lecture/Main Theorems and Concepts From Lecture**

**Random Variable (rv):** A numeric function $X : \Omega \rightarrow \mathbb{R}$ of the outcome.

**Range/Support:** The support/range of a random variable $X$, denoted $\Omega_X$, is the set of all possible values that $X$ can take on.

**Discrete Random Variable (drv):** A random variable taking on a countable (either finite or countably infinite) number of possible values.

**Probability Mass Function (pmf) for a discrete random variable $X$:** a function $p_X : \Omega_X \rightarrow [0,1]$ with $p_X(x) = P(X = x)$ that maps possible values of a discrete random variable to the probability of that value happening, such that $\sum_x p_X(x) = 1$.

**Expectation (expected value, mean, or average):** The expectation of a discrete random variable is defined to be $E[X] = \sum_x x p_X(x) = \sum_x xP(X = x)$. The expectation of a function of a discrete random variable $g(X)$ is $E[g(X)] = \sum_x g(x)p_X(x)$.

**Linearity of Expectation:** Let $X$ and $Y$ be random variables, and $a, b, c \in \mathbb{R}$. Then, $E[ax + by + c] = aE[X] + bE[Y] + c$.

**Exercises**

1. Suppose we have $N$ items in a bag, $K$ of which are successes. Suppose we draw (without replacement) until we have $k$ successes, $k \leq K \leq N$. Let $X$ be the number of draws until the $k^{th}$ success. What is $\Omega_X$? What is $p_X(n) = P(X = n)$? (We say $X$ is a “negative hypergeometric” random variable).

   $p_X(n) = P(X = n) = \frac{\Omega_X = \{k, k+1, \ldots, N-K+k\}}{\binom{k-1}{n-k}\binom{N-k}{K-(k-1)}\frac{N}{N-(n-1)}, n = k, k+1, \ldots, N-K+k}$

2. A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability $p_1$, to the left with probability $p_2$, and doesn’t move with probability $p_3$, where $p_1 + p_2 + p_3 = 1$. After 2 seconds, let $X$ be the location of the frog. Find the probability mass function for $X$, $p_X(k)$. Find $E[X]$. Find the probability mass function for $Y = |X|$, $p_Y(k)$, and $E[Y]$.

Let $L$ be a left step, $R$ be a right step, and $N$ be no step.

\[
P(X = -2) = P(LL) = p_2^2 \\
P(X = 2) = P(RR) = p_1^2 \\
P(X = 1) = P(RN \cup NR) = 2p_1p_3
\]
\[ P(X = -1) = P(LN \cup NL) = 2p_2p_3 \]
\[ P(X = 0) = P(NN \cup LR \cup RL) = p_3^2 + 2p_1p_2 \]

\[ p_X(k) = \begin{cases} 
  p_2^2, & k = -2 \\
  2p_2p_3, & k = -1 \\
  p_3^2 + 2p_1p_2, & k = 0 \\
  2p_1p_3, & k = 1 \\
  p_1^2, & k = 2 
\end{cases} \]

\[ E[X] = -2p_2^2 - 2p_2p_3 + 2p_1p_3 + 2p_1^2 \]

\[ p_Y(k) = \begin{cases} 
  p_3^2 + 2p_1p_2, & k = 0 \\
  2p_3(p_1 + p_2), & k = 1 \\
  p_1^2 + p_2^2, & k = 2 
\end{cases} \]

\[ E[Y] = 2p_3(p_1 + p_2) + 2(p_1^2 + p_2^2) \]

3. Suppose we have \( r \) independent random variables \( X_1, \ldots, X_r \) that each represent the number of coins flipped up to and including the first head, where \( P(\text{head}) = p \). Recall that each \( X_i \) has probability mass function,

\[ p_{X_i}(k) = P(X_i = k) = (1 - p)^{k-1}p \]

a) What do you think \( E[X_i] \) should be (without calculations) if \( p = \frac{1}{2} \)? If \( p = \frac{1}{3} \)? In the general case? (Proof in lecture next time.)

Should be 2, 3, and \( \frac{1}{p} \) in general.

b) Suppose we define \( X = X_1 + \cdots + X_r \). What does \( X \) represent, in English words? (Hint: think of performing each “trial” one after the other.)

The number of coins flipped up to and including the \( r^{th} \) head.

c) What is \( \Omega_X \)? Find the probability mass function for \( X, p_X(k) \).

\[ p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \ldots \]

d) Find \( E[X] \) using linearity of expectation.

\[ E[X] = E \left[ \sum_{i=1}^{r} X_i \right] = \sum_{i=1}^{r} E[X_i] = \sum_{i=1}^{r} \frac{1}{p} = \frac{r}{p} \]