1. Corrupted by their power, the judges running the popular game show America’s Next Top Mathematician have been taking bribes from many of the contestants. During each of two episodes, a given contestant is either allowed to stay on the show or is kicked off. If the contestant has been bribing the judges, she will be allowed to stay with probability 1. If the contestant has not been bribing the judges, she will be allowed to stay with probability $1/3$, independent of what happens in earlier episodes. Suppose that $1/4$ of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds.

(a) If you pick a random contestant, what is the probability that she is allowed to stay during the first episode?

Let $S_i$ be the event that she stayed during the $i$-th episode. By the Law of Total Probability,

$$P(S_1) = \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{2}$$

(b) If you pick a random contestant, what is the probability that she is allowed to stay during both episodes?

By the Law of Total Probability,

$$P(S_1 \cap S_2) = \frac{1}{4} \cdot 1 \cdot 1 + \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{3}$$

(c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

By the definition of conditional probability and the Law of Total Probability,

$$P(S_2 \mid S_1) = \frac{P(S_1 \cap S_2)}{P(S_1)} = \frac{\frac{1}{4} \cdot 1 \cdot 0 + \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{2}{3}}{\frac{1}{2}} = \frac{1/6}{1/2} = \frac{1}{3}$$

(d) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judges?

Let $B$ be the event that she bribed the judges. By Bayes’ Theorem,

$$P(B \mid S_1) = \frac{P(S_1 \mid B)P(B)}{P(S_1)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

2. A parallel system functions whenever at least one of its components works. Consider a parallel system of $n$ components and suppose that each component works with probability $p$ independently.
(a) If the system is functioning, what is the probability that component 1 is working?

\[
\frac{p}{1 - (1 - p)^n}
\]

(b) If the system is functioning and component 2 is working, what is the probability that component 1 is working?

\[p\]

3. A girl has 5 blue and 3 white marbles in her left pocket, and 4 blue and 4 white marbles in her right pocket. If she transfers a randomly chosen marble from left pocket to right pocket without looking, and then draws a randomly chosen marble from her right pocket, what is the probability that it is blue?

By the Law of Total Probability,

\[
\frac{5}{8} \cdot \frac{5}{9} + \frac{3}{8} \cdot \frac{4}{9} = \frac{37}{72}
\]

4. In a certain population, everyone is equally susceptible to colds. The number of colds suffered by each person during each winter season ranges from 0 to 4, with probability 0.2 for each value (see table below). A new cold prevention drug is introduced that, for people for whom the drug is effective, changes the probabilities as shown in the table. Unfortunately, the effects of the drug last only the duration of one winter season, and the drug is only effective in 20% of people, independently.

<table>
<thead>
<tr>
<th>number of colds</th>
<th>no drug or ineffective</th>
<th>drug effective</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
<td>0.4</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
<td>0.0</td>
</tr>
</tbody>
</table>

(a) Sneezy decides to take the drug. Given that he gets 1 cold that winter, what is the probability that the drug is effective for Sneezy?

Let \( E \) be the event that the drug is effective for Sneezy, and \( C_i \) be the event that he gets \( i \) colds the first winter. By Bayes’ Theorem,

\[
P(E \mid C_1) = \frac{P(C_1 \mid E)P(E)}{P(C_1 \mid E)P(E) + P(C_1 \mid \overline{E})P(\overline{E})} = \frac{0.3 \times 0.2}{0.3 \times 0.2 + 0.2 \times 0.8} = \frac{3}{11}
\]

(b) The next year he takes the drug again. Given that he gets 2 colds in this winter, what is the updated probability that the drug is effective for Sneezy?

Let the reduced sample space for part (b) be \( C_1 \) from part (a). Let \( D_i \) be the event that he gets \( i \) colds the second winter. By Bayes’ Theorem,

\[
P(E \mid D_2) = \frac{P(D_2 \mid E)P(E)}{P(D_2 \mid E)P(E) + P(D_2 \mid \overline{E})P(\overline{E})} = \frac{0.2 \times \frac{3}{11}}{0.2 \times \frac{3}{11} + 0.2 \times \frac{8}{11}} = \frac{3}{11}
\]

2
(c) The third winter he decides not to bother taking the drug and gets 2 colds. He argues that the drug must not have been effective for him, since he got the same number of colds last year as this year. Comment on his logic.

The posterior probability that the drug is effective is $\frac{3}{11}$. This is greater than the prior probability $\frac{1}{5}$, so the drug probably was effective.

5. Guildenstern has three coins $C_1, C_2, C_3$ in a bag. $C_1$ has $P(\text{heads}) = 1$, $C_2$ has $P(\text{heads}) = 0$, and $C_3$ has $P(\text{heads}) = p$. He takes a random coin from the bag, each coin equally probable, and flips this same coin some number of times.

(a) Suppose $q$ is the conditional probability that he flipped coin $C_1$, given that the flip came up heads. Determine $p$ as a function of $q$.

Let $F_i$ be the event that he flipped coin $C_i$ and $H$ be the event that the flip came up heads. By Bayes’ Theorem,

$$q = P(F_1 | H) = \frac{P(H \cap F_1)}{P(H)} = \frac{P(H | F_1)P(F_1)}{P(H | F_1)P(F_1) + P(H | F_2)P(F_2) + P(H | F_3)P(F_3)}$$

$$= \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + 0 \times \frac{1}{3} + p \times \frac{1}{3}}$$

$$= \frac{1}{1 + p}$$

$$p = \frac{1}{q} - 1$$

(b) What is the probability that the first $n$ flips come up tails?

Let $T$ be the event that he flips $n$ tails in a row. By the Law of Total Probability,

$$P(T) = P(T | F_1)P(F_1) + P(T | F_2)P(F_2) + P(T | F_3)P(F_3)$$

$$= 0 \times \frac{1}{3} + 1 \times \frac{1}{3} + (1 - p)^n \times \frac{1}{3}$$

$$= \frac{1}{3}(1 + (1 - p)^n)$$

(c) Given that the first $n$ flips come up tails, what is the probability he flipped $C_1$? $C_2$? $C_3$?

By Bayes’ Theorem,

$$P(F_1 | T) = \frac{P(T | F_1)P(F_1)}{P(T)}$$

$$= \frac{0 \times \frac{1}{3}}{\frac{1}{3}(1 + (1 - p)^n)}$$

$$= 0$$


\[ P(F_2 \mid T) = \frac{P(T \mid F_2)P(F_2)}{P(T)} = \frac{1 \times \frac{1}{3}}{\frac{1}{3}(1 + (1 - p)^n)} = \frac{1}{1 + (1 - p)^n} \]

\[ P(F_3 \mid T) = \frac{P(T \mid F_3)P(F_3)}{P(T)} = \frac{(1 - p)^n \times \frac{1}{3}}{\frac{1}{3}(1 + (1 - p)^n)} = \frac{(1 - p)^n}{1 + (1 - p)^n} \]

6. Guildenstern has a fair coin and a “magic” coin that comes up heads with probability \( p_1 > \frac{1}{2} \). Suppose he picks a coin at random, with probability \( p_2 \) of choosing the magic coin and \( 1 - p_2 \) of choosing the fair coin, and tosses it \( n \) times. All of the tosses come up heads. He would like to convince Rosencrantz that he flipped the magic coin. Rosencrantz only believes him if the conditional probability that it is the magic coin, given the \( n \) heads, is at least 99%. Derive a function \( n = f(p_1, p_2) \) that gives the minimum number of consecutive heads \( n \) to convince Rosencrantz that Guildenstern flipped the magic coin. Remember that \( n \) must be a positive integer.

Let \( M \) be the event that Guildenstern picked the magic coin, and \( H \) be the event that he flipped \( n \) heads in a row. By Bayes’ Theorem,

\[
P(M \mid H) = \frac{P(H \mid M)P(M)}{P(H \mid M)P(M) + P(H \mid \overline{M})P(\overline{M})} = \frac{p_1^n p_2}{p_1^n p_2 + (\frac{1}{2})^n(1 - p_2)} \geq 0.99
\]

\[
0.01 p_1^n p_2 \geq 0.99(1 - p_2)/2^n
\]

\[
(2p_1)^n \geq 99 \cdot \frac{1 - p_2}{p_2} = \frac{99}{p_2} - 99
\]

\[
n = \left\lceil \frac{\log(\frac{99}{p_2} - 99)}{\log(2p_1)} \right\rceil
\]

7. This problem demonstrates that independence can be “broken” by conditioning. Let \( D_1 \) and \( D_2 \) be the outcomes of two independent rolls of a fair die. Let \( E \) be the event “\( D_1 = 1 \)”, \( F \) be the event “\( D_2 = 6 \)”, and \( G \) be the event “\( D_1 + D_2 = 7 \)”. Even though \( E \) and \( F \) are independent, show that

\[
P(E \cap F \mid G) \neq P(E \mid G)P(F \mid G).
\]
\[ \begin{align*}
P(E \mid G) &= P(D_1 = 1 \mid D_1 + D_2 = 7) = 1/6 \\
P(F \mid G) &= P(D_2 = 6 \mid D_1 + D_2 = 7) = 1/6 \\
P(E \cap F \mid G) &= P(D_1 = 1 \cap D_2 = 6 \mid D_1 + D_2 = 7) = 1/6
\end{align*}\]