CSE 312: Foundations of Computing II
Quiz Section #2: Binomial Theorem, Pigeonhole Principle, Introduction to Probability

<u>Review/Mini-Lecture/Main Theorems and Concepts From Lecture</u>

Binomial Theorem:

Principle of Inclusion-Exclusion (PIE):

Pigeonhole Principle: If there are n pigeons with k holes and $n > k$, then at least one hole contains at least ___ pigeons.

Complementary Counting (Complementing): If asked to find the number of ways to do X, you can:

Definitions

Sample Space: The set of all possible outcomes of an experiment, denoted Ω or S
Event: Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$
Union: The union of two events E and F is denoted $E \cup F$
Intersection: The intersection of two events E and F is denoted $E \cap F$ or EF
Mutually Exclusive: Events E and F are mutually exclusive iff $E \cap F = \emptyset$
Complement: The complement of an event E is denoted E^C or $\bar{E}$ or $\neg E$, and is equal to $\Omega \setminus E$
DeMorgan's Laws: $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$
Probability of an event E: denoted $P(E)$ or $\Pr(E)$ or $\mathbb{P}(E)$
Partition: Nonempty events $E_1, \dots, E_n$ partition the sample space Ω iff

- $E_1, \dots, E_n$ are exhaustive: ____ , and
- $E_1, \dots, E_n$ are pairwise mutually exclusive: ____
  - Note that for any event A (with $A \neq \emptyset, A \neq \Omega$): _______ and _______ partition Ω

Axioms of Probability and their Consequences

1. (Non-negativity) For any event E, $P(E) \geq$ _____
2. (Normalization) $P(\Omega) =$ ____
3. (Additivity) If E and F are mutually exclusive, then $P(E \cup F) =$ ________________________

- $P(E) + P(E^C) =$ ____
- If $E \subseteq F$, $P(E)$ ____ $P(F)$
- $P(E \cup F) =$ ____

Equally Likely Outcomes: If we have equally likely outcomes in finite sample space Ω, and E is an event, then

$$P(E) = \underline{\hspace{6cm}}$$

- Note: Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn't matter in both.

Exercises

1. Give a **combinatorial** proof that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$. Do **not** use the binomial theorem. (Hint: you can count the number of subsets of $[n] = \{1,2,\dots,n\}$). Note: A combinatorial proof is one in which you explain how to count something in two different ways – then those formulae must be equivalent if they both indeed count the same thing.

2. How many ways are there to have three initials (upper case letters) that have two being the same or all three being the same?

3. Suppose there are N items in a bag, with K of them marked as successes in total (and the rest are marked as failures). We draw n of them, without replacement. Each item is equally likely to be drawn. Let X be the number of successes we draw (out of n). What is $P(X = k)$, that is, the probability we draw exactly k successes?

4. Suppose we have 12 chairs (in a row) with 9 TA's, and Professors Ruzzo, Karlin, and Tompa to be seated. Suppose all seatings are equally likely. What is the probability that every professor has a TA to his/her immediate left and right?

5. Suppose Joe is a k-legged robot, who wears a sock and a shoe on each leg. Suppose he puts on k socks and k shoes in some order, each equally likely. In how many ways can he put on his socks and shoes in a valid order? We say an ordering is valid if, for every leg, the sock gets put on before the shoe. Assume all socks are indistinguishable from each other, and shoes are indistinguishable from each other.

6. Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent. Repeat the question for the letters "AAAAABBB".