CSE 312: Foundations of Computing II
Quiz Section #2: Binomial Theorem, Pigeonhole Principle, Introduction to Probability

Review/Mini-Lecture/Main Theorems and Concepts From Lecture

Binomial Theorem: \(\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}, (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\).

Principle of Inclusion-Exclusion (PIE): 2 events: \(|A \cup B| = |A| + |B| - |A \cap B|\)
3 events: \(|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|\)
In general: + singles – doubles + triples – quads + …

Pigeonhole Principle: If there are \(n\) pigeons with \(k\) holes and \(n > k\), then at least one hole contains at least 2 pigeons.

Complementary Counting (Complementing): If asked to find the number of ways to do X, you can:
find the total number of ways and then subtract the number of ways to not do X.

Definitions

Sample Space: The set of all possible outcomes of an experiment, denoted \(\Omega\) or \(S\)
Event: Some subset of the sample space, usually a capital letter such as \(E \subseteq \Omega\)
Union: The union of two events \(E\) and \(F\) is denoted \(E \cup F\)
Intersection: The intersection of two events \(E\) and \(F\) is denoted \(E \cap F\) or \(EF\)
Mutually Exclusive: Events \(E\) and \(F\) are mutually exclusive iff \(E \cap F = \emptyset\)
Complement: The complement of an event \(E\) is denoted \(E^C\) or \(\bar{E}\) or \(\neg E\), and is equal to \(\Omega \setminus E\)
DeMorgan’s Laws: \((E \cup F)^C = E^C \cap F^C\) and \((E \cap F)^C = E^C \cup F^C\)
Probability of an event \(E\): denoted \(P(E)\) or \(\Pr(E)\) or \(\mathbb{P}(E)\)
Partition: Nonempty events \(E_1, \ldots, E_n\) partition the sample space \(\Omega\) iff
- \(E_1, \ldots, E_n\) are exhaustive: \(E_1 \cup E_2 \cup \ldots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega\), and
- \(E_1, \ldots, E_n\) are pairwise mutually exclusive: \(\forall i \neq j, E_i \cap E_j = \emptyset\)
  - Note that for any event \(A\) (with \(A \neq \emptyset, A \neq \Omega\)): \(A\) and \(A^C\) partition \(\Omega\)

Axioms of Probability and their Consequences

1. (Non-negativity) For any event \(E\), \(P(E) \geq 0\)
2. (Normalization) \(P(\Omega) = 1\)
3. (Additivity) If \(E\) and \(F\) are mutually exclusive, then \(P(E \cup F) = P(E) + P(F)\)
   - \(P(E) + P(E^C) = 1\)
   - If \(E \subseteq F\), \(P(E) \leq P(F)\)
   - \(P(E \cup F) = P(E) + P(F) - P(E \cap F)\)

Equally Likely Outcomes: If we have equally likely outcomes in finite sample space \(\Omega\), and \(E\) is an event, then \(P(E) = \frac{|E|}{|\Omega|}\).
• Note: Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn’t matter in both.

**Exercises**

1. Give a **combinatorial** proof that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Do **not** use the binomial theorem. (Hint: you can count the number of subsets of $[n] = \{1,2, ..., n\}$). Note: A combinatorial proof is one in which you explain how to count something in two different ways – then those formulae must be equivalent if they both indeed count the same thing.

   Fix a subset of $[n]$ of size $k$. There are $\binom{n}{k}$ such subsets because we choose any $k$ elements out of the $n$, with order not mattering since these are sets. Subsets can be of size $k = 0,1, ..., n$. So the total number of subsets of $[n]$ is $\sum_{k=0}^{n} \binom{n}{k}$. On the other hand, each element of $[n]$ is either in a subset or not. So there are 2 possibilities for the first element (in or out), 2 for the second, etc. So there are $2^n$ subsets of $[n]$. Therefore, $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Note that this agrees with the binomial theorem using $x = y = 1$.

2. How many ways are there to choose three initials (upper case letters) that have two being the same or all three being the same?

   **Complementary counting.** Count the total $26^3$ and subtract the number with all distinct initials $26 \times 25 \times 24$ to get $26^3 - 26 \times 25 \times 24$.

3. Suppose there are $N$ items in a bag, with $K$ of them marked as successes in total (and the rest are marked as failures). We draw $n$ of them, **without** replacement. Each item is equally likely to be drawn. Let $X$ be the number of successes we draw (out of $n$). What is $P(X = k)$, that is, the probability we draw exactly $k$ successes?

   $$P(X = k) = \binom{K}{k} \frac{\binom{N-K}{n-k}}{\binom{N}{n}}$$

   We choose $k$ out of the $K$ successes, and $n - k$ out of the $N - K$ failures. The denominator is the total number of ways to choose $n$ objects out of $N$.

4. Suppose we have 12 chairs (in a row) with 9 TA’s, and Professors Ruzzo, Karlin, and Tompa to be seated. Suppose all seatings are equally likely. What is the probability that every professor has a TA to his/her immediate left and right?

   Imagine we permute all 9 TA’s first – there are $9!$ ways to do this. Then, there are 8 spots between them, in which we pick 3 for the Professors to sit – order matters since each Professor is distinct. So the total ways is $P(8,3) \times 9!$.

   The total number of ways to seat all 12 of us is simply $12!$.

   The probability is then $\frac{P(8,3) \times 9!}{12!}$.
5. Suppose Joe is a k-legged robot, who wears a sock and a shoe on each leg. Suppose he puts on k socks and k shoes in some order, each equally likely. In how many ways can he put on his socks and shoes in a valid order? We say an ordering is valid if, for every leg, the sock gets put on before the shoe. Assume all socks are indistinguishable from each other, and shoes are indistinguishable from each other.

First, note there are 2k objects – k shoes and k socks. Suppose we describe a sequence of actions, 
\[ \text{Sock}_1, \text{Shoe}_1, \text{Sock}_2, \text{Shoe}_2, \ldots, \text{Sock}_k, \text{Shoe}_k \]
This particular example means we first put a sock on leg 1, then a shoe on leg 1, then a sock on leg 2, etc. There are \((2k)!\) ways to order these objects – however, for each pair, this is only one valid ordering – the sock must come before the shoe. So we divide by \(2^k\). So the total number of ways is \(\frac{(2k)!}{2^k}\).

Alternatively, \(P(\text{valid ordering}) = \frac{|\text{valid orderings}|}{|\text{orderings}|}\) so that \(|\text{valid orderings}| = P(\text{valid ordering}) \times |\text{orderings}|\). We can compute \(P(\text{valid ordering}) = \left(\frac{1}{2}\right)^k\). Notice for any sequence of actions with each equally likely, the probability that the sock came before the shoe on a particular leg is \(\frac{1}{2}\), so the probability this happened for each leg is \(\left(\frac{1}{2}\right)^k\). Then \(|\text{orderings}| = (2k)!\) because there are \(2k\) actions that we can permute, all distinct. Multiplication gives the same answer as above.

6. Find the number of ways to rearrange the word “INGREDIENT”, such that no two identical letters are adjacent to each other. For example, “INGREEDINT” is invalid because the two E’s are adjacent. Repeat the question for the letters “AAAAABBB”.

We use inclusion-exclusion. Let \(\Omega\) be the set of all anagrams (permutations) of “INGREDIENT”, and \(A_I\) be the set of all anagrams with two consecutive I’s. Define \(A_E\) and \(A_N\) similarly. \(A_I \cup A_E \cup A_N\) clearly are the set of anagrams we don’t want. So we use complementing to count the size of \(\Omega \setminus (A_I \cup A_E \cup A_N)\). By inclusion exclusion, \(|A_I \cup A_E \cup A_N| = \text{singles} - \text{doubles} + \text{triples}\), and by complementing, \(|\Omega \setminus (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|\).

First, \(|\Omega| = \frac{10!}{2!2!2!2!2!}\) because there are 2 of each of \(I, E, N\)’s (multinomial coefficient). Clearly, the size of \(A_I\) is the same as \(A_E\) and \(A_N\). So \(|A_I| = \frac{9!}{2!2!}\) because we treat the two adjacent I’s as one entity. We also need \(|A_I \cap A_E| = \frac{8!}{2!}\) because we treat the two adjacent I’s as one entity and the two adjacent E’s as one entity (same for all doubles). Finally, \(|A_I \cap A_E \cap A_N| = 7!\) since we treat each pair of adjacent I’s, E’s, and N’s as one entity.

Putting this together gives \(\frac{10!}{2!2!2!2!2!} - \left(\binom{3}{1} \cdot \frac{9!}{2!2!} - \binom{3}{2} \cdot \frac{8!}{2!}\right) + \binom{3}{3}7!\).

For the second question, note that no A’s and no B’s can be adjacent. So let us put the B’s down first:
\[ _B _B _B _ \]
By the pigeonhole principle, two A’s must go in the same slot, but then they would be adjacent, so there are no ways.