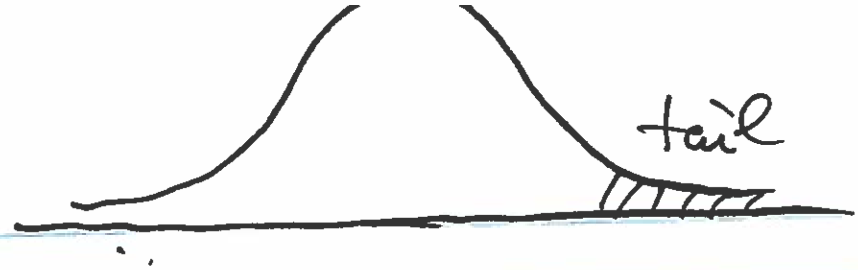
Tail Bounds



(1) **Markov's Inequality:**

Theorem: If X is a nonnegative random variable, then for any $\alpha > 0$,
$P(X \geq \alpha) \leq E[X]/\alpha$. Equivalently,
$P(X \geq kE[X]) \leq 1/k$.

Proof: $E[X] = \sum_x x p(x) = \sum_{x<\alpha} x p(x) + \sum_{x \geq \alpha} x p(x)$

$\geq 0 + \sum_{x \geq \alpha} \alpha p(x) = \alpha \sum_{x \geq \alpha} p(x) = \alpha P(X \geq \alpha)$

Ex: Suppose your expected daily business cost is $E[X] = 1500$, where X is the daily business cost. What is $P(X > 6000)$?
$P(X > 6000) = P(X > 4E[X]) \leq 1/4$.
$P(X > 6000) \leq P(X \geq 6000) \leq 1/4$.

(2) **Chebyshev's Inequality:**

Theorem: If Y is a r.v. with $E[Y] = \mu$ and $Var(Y) = \sigma^2$, then for any $\alpha > 0$,
$P(|Y - \mu| \geq \alpha) \leq \sigma^2/\alpha^2$. Equivalently,
$P(|Y - \mu| \geq k\sigma) \leq 1/k^2$.

Proof: Let $X = (Y - \mu)^2$. X is nonnegative,
so $P(|Y - \mu| \geq \alpha) = P(X \geq \alpha^2) \leq E[X]/\alpha^2$
$= Var(Y)/\alpha^2 = \sigma^2/\alpha^2$.

$a \geq b$ iff $a^2 \geq b^2$ if $a, b \geq 0$

Ex: Same example: $\mu = E[X] = 1500$, $\sigma = 500$.
$P(X \geq 6000) = P(X - 1500 \geq 4500) \leq P(|X - 1500| \geq 4500)$
$= P(|X - E[X]| \geq 9\sigma) \leq 1/9^2 = 1/81$.

$\mu = 1500$, $\sigma = 500$

$P(X \geq 2000) = P(X - 1500 \geq 500) \leq P(|X-1500| \geq 500)$
$\leq \cdots$

(3) Chernoff Bound:

<u>Theorem</u>: Suppose $X \sim Bin(n, p)$ and $\mu = E[X] = np$.

$P(X \geq (1+\delta)\mu) \leq e^{-\frac{1}{3}\delta^2 \mu}$, for any $0 < \delta < 1$,

$P(X \leq (1-\delta)\mu) \leq e^{-\frac{1}{2}\delta^2 \mu}$, for any $0 < \delta < 1$.