Defn.: R.V.'s X and Y are independent iff $(\forall x)(\forall y)\ P(X=x \,\&\, Y=y) = P(X=x)P(Y=y)$

Dependent indicator r.v.'s $X_1, X_2, X_3 \ldots$

$P(X_i = 1)$. Shuffle 4 aces and lay face-down in 4 piles.

For $i \in \{1,2,3,4\}$,

Let $X_i = \begin{cases} 1, & \text{if } i^{\text{th}} \text{ pile has a spade} \\ 0, & \text{otherwise} \end{cases}$

$P(X_i=1) = 1/4$, even though X_i and X_j are dependent.

$$P(X_2=1) = P(X_2=1 \mid X_1=1)P(X_1=1) + P(X_2=1 \mid X_1=0)P(X_1=0)$$
$$= \cancel{0 \cdot \tfrac{1}{4}} + \tfrac{1}{3} \cdot \tfrac{3}{4} = \tfrac{1}{4}$$

$X = X_1 + X_2 + X_3 + X_4$ = # of spades in all 4 piles

$$\begin{aligned} E[X] &= E[X_1 + X_2 + X_3 + X_4] \\ &= E[X_1] + E[X_2] + E[X_3] + E[X_4] \\ &= P(X_1=1) + P(X_2=1) + P(X_3=1) + P(X_4=1) \\ &= 4 \cdot \tfrac{1}{4} = 1. \end{aligned}$$

Ex: Flip a fair coin $2n$ times independently.

Let X = # heads in first n flips,

Y = # heads in last n flips, and

Z = # heads in all $2n$ flips.

X and Y are independent.

X and Z are dependent.

$P(X=0) > 0$ and $P(Z=n+1) > 0$, but

$P(X=0 \,\&\, Z=n+1) = 0$.

__Theorem:__ If X and Y are independent r.v.'s, then $E[XY]=E[X]E[Y]$.

__Proof:__ $E[XY] = \sum_x \sum_y xy\, P(X=x \,\&\, Y=y)$

$= \sum_x \sum_y xy\, P(X=x)P(Y=y)$ (ind.)

$= \left(\sum_x x P(X=x)\right)\left(\sum_y y P(Y=y)\right)$

$= E[X]E[Y]$

__Theorem:__ If X and Y are independent r.v.'s, then $Var(X+Y) = Var(X) + Var(Y)$.

__Proof:__ $Var(X+Y) = E[(X+Y)^2] - (E[X+Y])^2$

$= E[X^2+2XY+Y^2] - (E[X]+E[Y])^2$ (lin.)

$= E[X^2]+2E[XY]+E[Y^2]$ (lin.)

$\quad - (E[X])^2 - 2E[X]E[Y] - (E[Y])^2$

$= (E[X^2]-(E[X])^2) + (E[Y^2]-(E[Y])^2)$

$\quad + 2E[X]E[Y] - 2E[X]E[Y]$ (ind.)

$= Var(X) + Var(Y)$.

Some important specific discrete r.v.'s.

__Defn:__ X is uniform on $[a,b]$, denoted $X \sim Unif(a,b)$ if X is equally likely to be any integer in $[a,b]$.

PMF: $p(x) = \frac{1}{b-a+1}$, for $x \in \{a, a+1, \ldots, b\}$

$\quad = 0$, otherwise

$E[X] = \frac{1}{2}(b+a)$, $Var(X) = \frac{1}{12}(b-a)(b-a+2)$