Variance

Defn: Let X be a r.v. with $E[X]=\mu$. The **variance** of X is $Var(X) = E[(X-\mu)^2]$, often denoted σ^2.

Defn: The **standard deviation** of X is $\sigma = \sqrt{Var(X)}$.

In the \$1 (X) and \$1000 (Y) games:

$$Var(X) = E[(X-\mu_X)^2] = E[X^2] = 1$$

$$Var(Y) = E[(Y-\mu_Y)^2] = E[Y^2] = 1{,}000{,}000.$$

Theorem: $Var(X) = E[X^2] - (E[X])^2$.

Proof: $Var(X) = E[(X-\mu)^2]$, where $\mu = E[X]$

$$= \sum_x (x-\mu)^2 p(x) = \sum_x (x^2 - 2\mu x + \mu^2) p(x)$$

$$= \sum_x x^2 p(x) - \sum_x 2\mu x p(x) + \sum_x \mu^2 p(x)$$

$$= \sum_x x^2 p(x) - 2\mu \sum_x x p(x) + \mu^2 \sum_x p(x)$$

$$= E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - \mu^2$$

$$= E[X^2] - (E[X])^2$$

Theorem: $Var(aX+b) = a^2 Var(X)$, where a,b const.

Proof: Let $\mu = E[X]$.

$$Var(aX+b) = E[((aX+b)-(a\mu+b))^2]$$

$$= E[(a(X-\mu))^2] = E[a^2(X-\mu)^2]$$

$$= a^2 E[(X-\mu)^2] = a^2 Var(X).$$

In general, $Var(X+Y) \neq Var(X) + Var(Y)$

E.g., $Var(X+X) = Var(2X) = 4Var(X)$

$$\neq Var(X) + Var(X)$$