Linearity of Expectation

Defn: If \( X \) is a random variable and \( g \) is a function, \( E[g(X)] = \sum_x g(x)p(x) \)

Theorem: For any constants \( a \) and \( b \),
\[ E[aX+b] = aE[X]+b. \]

Proof: \[ E[aX+b] = \sum_x (aX+b)p(x) \]
\[ = a \sum_x xp(x) + b \sum_x p(x) = aE[X] + b \]

Ex: You pay $1 to play the following game. A fair coin is flipped up to and including the first head, and you are paid 12$ per flip. Do you expect to win or lose money?

Let \( X \) be the number of flips until the first head.
Your expected gain is \( E[12X-100] = 12E[X]-100 \)
\[ = 12 \cdot 8 - 100 = -4 \] because \( E[X] = \frac{1}{1/2} = 8 \)

Theorem: Let \( X \) and \( Y \) be two random variables, possibly dependent. Then \( E[X+Y] = E[X] + E[Y] \).

Proof: Let \( (X(s), Y(s)) \) be the values of \( X, Y \) for some \( s \in \Omega \).
\[ E[X+Y] = \sum_{s \in \Omega} (X(s)+Y(s))p(s) = \sum_{s \in \Omega} X(s)p(s) + \sum_{s \in \Omega} Y(s)p(s) \]
\[ = E[X] + E[Y] \]
Ex: Let $X$ be the number of heads when a coin with $P(\text{Heads}) = p$ is flipped $n$ times independently.

Let $X_i = 1$, if $i$th flip is heads, for $1 \leq i \leq n$, 0 otherwise.

"Indicator random variables" only have values 0 or 1.

\[
E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = P(i\text{-th flip is Heads})
\]

\[
X = \sum_{i=1}^n X_i
\]

\[
E[X] = E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n p = np
\]

Linearity is special. In general, $E[XY] \neq E[X]E[Y]$

$E[X^2] \neq (E[X])^2$

$E[\sqrt{X}] \neq \sqrt{E[X]}$

$E(X!) \neq (E[X])!$

Variance