Random Variables

Defn: A random variable is a numeric function of the outcome.

Ex: 1. Number of heads when a coin is flipped 20 times.
2. Total of 2 die rolls.
3. Number of coin flips until first head.

Suppose coin has probability p of heads. Let X be the number of flips up to and including the first head. X is a r.v.

\[ P(X=1) = P(TH) = p \]
\[ P(X=2) = P(TTH) = p(1-p) \]
\[ P(X=i) = P(T^{i-1}H) = p(1-p)^{i-1} \]

Defn: If a random variable has a countable number of possible values, it is called discrete.

Defn: If X is a discrete r.v. with values in some countable set T, the probability mass function (pmf) of X is

\[ p(a) = \Pr(X=a) = \sum_{a \in T} P(X=a), \text{ if } a \in T \]
\[ 0, \text{ otherwise.} \]

Note: \( \sum_{a \in T} p(a) = 1 \)
Defn: For a discrete r.v. X with pmf \( p \),
the expectation (or expected value or mean) of \( X \) is
\[
E[X] = \sum_{x} x p(x).
\]

In the case of equally probable outcomes,
\( E[X] \) is just the average value. But in general, it's a weighted average, each \( x \) weighted by its probability \( p(x) \).

For the binomial random variable of slide 13
(# of heads in \( n \) ind. coin flips)
\[
E[X] = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = ?
\]

Back to geometric random variable.
Let \( X \) be # of coin flips up to and including first head of a coin with prob \( p \) of heads.

pmf of \( X \): \( p(i) = p (1-p)^{i-1} \)
\[
E[X] = \sum_{i=1}^{\infty} i p(i) = \sum_{i=1}^{\infty} i p (1-p)^{i-1} = p \sum_{i=1}^{\infty} i (1-p)^{i-1}
\]

\[
\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}, \text{ if } |r|<1
\]

Take \( \frac{d}{dr} \) of both sides:
\[
\sum_{i=0}^{\infty} i r^{i-1} = \frac{1}{(1-r)^2}
\]

\[
E[X] = p \sum_{i=1}^{\infty} i (1-p)^{i-1} = p \cdot \frac{1}{(1-(1-p))^2} = \frac{1}{p}
\]
Ex: if \( p = \frac{1}{2}, \) then \( \mathbb{E}[X] = 2 \)
if \( p = \frac{1}{10}, \) then \( \mathbb{E}[X] = 10 \)