Independent Events

Defn: Two events E and F are independent iff $P(E \cap F) = P(E)P(F)$. Otherwise, E and F are dependent.

Ex: Roll 2 fair dice, yielding values D_1 and D_2. Let E be "$D_1 = 1$" and F be "$D_1 + D_2 = 7$" and G be "$D_1 + D_2 = 5$".

$P(E) = \frac{1}{6}$, $P(F) = \frac{6}{36} = \frac{1}{6}$

$P(E \cap F) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P(E)P(F)$, so E and F are independent.

$P(G) = \frac{4}{36} = \frac{1}{9}$

$P(E \cap G) = \frac{1}{36} \neq P(E)P(G) = \frac{1}{6} \cdot \frac{1}{9} = \frac{1}{54}$

Theorem: If $P(F) > 0$, then E and F are independent iff $P(E|F) = P(E)$.

Proof: $\Rightarrow$: Suppose E and F are independent.

$P(E \cap F) = P(E)P(F)$. (defn of ind.)

$P(E \cap F) = P(E|F)P(F)$ (chain rule)

$P(E)P(F) = P(E|F)P(F)$

$P(E) = P(E|F)$

$\Leftarrow$: $P(E|F) = P(E)$

$P(E \cap F) = P(E|F)P(F) = P(E)P(F)$ (chain rule)

E and F are independent (defn of ind)

<u>Defn</u>: $E_1, E_2, \ldots, E_n$ are independent iff for every subset S of $\{1, 2, \ldots, n\}$,

$$P\left(\bigcap_{i \in S} E_i\right) = \prod_{i \in S} P(E_i).$$

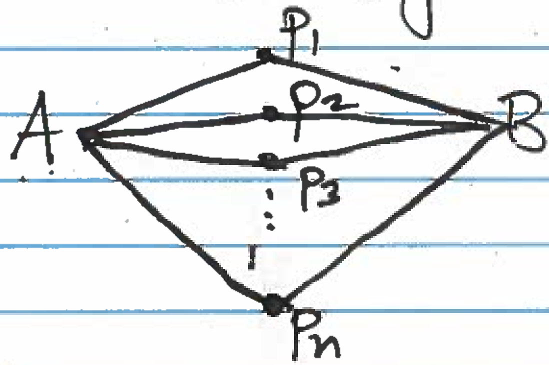
<u>Ex</u>: Suppose a biased coin comes up heads with prob p and tails with prob $1-p$. Suppose it is flipped n times independently.

$P(n \text{ heads}) = p^n$

$P(\text{first } k \text{ heads and remainder tails}) = p^k(1-p)^{n-k}$

<u>Ex</u>: Network failures:

Suppose we have n routers in parallel, where i^{th} router fails with probability p_i, independent of the others. That is, all n failures are independent.



$$
\begin{aligned}
&P(\text{A can communicate with B}) = \\
&\quad 1 - P(\text{all } n \text{ routers fail}) \\
&= 1 - \prod_{i=1}^{n} P(i^{th} \text{ router fails}) \qquad \text{(independence)} \\
&= 1 - \prod_{i=1}^{n} p_i = 1 - p_1 p_2, \cdots p_n
\end{aligned}
$$

Suppose these n routers are in series.

$$
\begin{aligned}
&P(\text{A can communicate with B}) \\
&= P(\text{no router fails}) \\
&= (1-p_1)(1-p_2)\cdots(1-p_n) \qquad \text{(independence)} \\
&= \prod_{i=1}^{n} (1-p_i)
\end{aligned}
$$