Bayes' Theorem (Rev. Thomas Bayes, c. 1701-1761):
\[ P(F|E) = \frac{P(E|F)P(F)}{P(E)} \]

Proof:
\[ P(F|E) = \frac{P(E|F)P(F)}{P(E)} = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|\bar{F})P(F)} \]

Corollary:
\[ P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|\bar{F})P(F)} \]

Proof: LTP

Why it's useful: Reverses conditioning
Example: \( F = \) disease, \( E = \) test result

You might know \( P(E|F) \)
Want to know \( P(F|E) \) if you had a + test result.

Example:
- 60% of e-mail is spam
- 90% of spam has a forged header
- 20% of nonspam has a forged header.

Let \( F = \) forged header, \( J = \) spam

What is \( P(J|F) \)?

\[
P(J|F) = \frac{P(F|J)P(J)}{P(F|J)P(J) + P(F|\bar{J})P(J)}
\]

\[ = \frac{0.9 \times 0.6}{0.9 \times 0.6 + 0.2 \times (1-0.6)} \approx 0.871 \]

Prior probability = 0.6

Posterior probability = 0.871
Ex: Paternity testing

Child has \((A,a)\) pair for some gene \((A \neq B, a)\)

Mother has \((A,A)\)

2 possible fathers, \(F_1 = (a,a)\), \(F_2 = (A,a)\).

\[ P(F_1) = p, \quad P(F_2) = 1 - p \]

\[ P(F_1 | B_{aa}) = \frac{P(B_{aa} | F_1) P(F_1)}{P(B_{aa} | F_1) P(F_1) + P(B_{aa} | F_2) P(F_2)} \]

\[ = \frac{1 \cdot p}{1 \cdot p + \frac{1}{2} (1-p)} = \frac{2p}{2p + 1 - p} = \frac{2p}{1+p} \]

\[ \geq \frac{2p}{1+p} \text{ posterior} \]

For example, if \( p = \frac{1}{4} \), then \( \frac{2p}{1+p} = \frac{2}{3} \)