Outcomes are not necessarily equiprobable

Ex: Let $\Omega = \{0, 1, 2\}$, the number of heads in 2 fair coin flips. $P(0) = P(2) = 1/4$, but $P(1) = 1/2$.

General defn of conditional probability is

If $P(F) \neq 0$, then $P(E|F) = \dfrac{P(E \cap F)}{P(F)}$.

Ex. Let E be the event "2 heads" and F be the event "≥ 1 head".

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)}{P(F)} = \frac{1/4}{1/2 + 1/4} = \frac{1}{3}.$$

Chain rule: $P(E \cap F) = P(E|F)P(F)$.

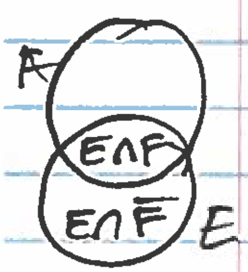
Generalized:

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1, E_2)\cdots \cdots P(E_n|E_1, E_2, \ldots, E_{n-1})$$

Law of Total Probability

If E and F are events, then

$$P(E) = P(E|F)P(F) + P(E|\bar{F})P(\bar{F})$$

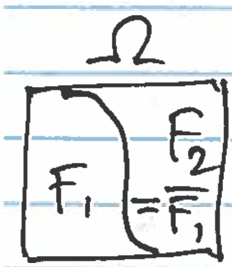


Proof: $P(E) = P((E \cap F) \cup (E \cap \bar{F}))$

$= P(E \cap F) + P(E \cap \bar{F})$

$= P(E|F)P(F) + P(E|\bar{F})P(\bar{F})$

$[= P(E|F)P(F) + P(E|\bar{F})(1 - P(F))]$



More generally, if $F_1, F_2, \ldots, F_n$ partition Ω, then

$$P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i).$$

Ex: Sally will take either CSE331 or CSE351. She will get an A in 331 with probability 3/4 and an A in 351 with probability 3/5. She flips a fair coin to decide which to take.

$$P(A) = P(A|331)P(331) + P(A|351)P(351)$$
$$= \frac{3}{4} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{2} = \frac{27}{40}.$$

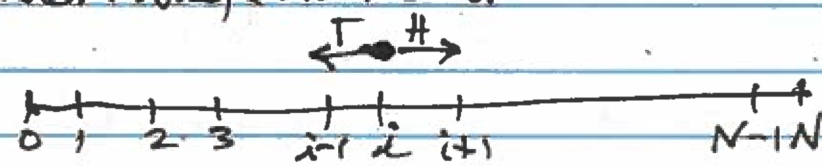
Gamber's Ruin

A has \$$i$, B has \$$(N-i)$.

Flip a fair coin: if H, B pays A \$1, if T, A pays B \$1.

~~Whose~~ Whoever get all \$$N$ wins.

Random walk on the line:



Let E_i = event that A wins starting with \$$i$.

What is $P(E_i)$?

Condition on first coin flip using LTP.

$$p_i = P(E_i) = P(E_i|H)P(H) + P(E_i|T)P(T)$$
$$p_i = \tfrac{1}{2}(p_{i+1} + p_{i-1})$$
$$2p_i = p_{i+1} + p_{i-1}$$
$$p_{i+1} - p_i = p_i - p_{i-1}$$
$$p_2 - p_1 = p_1 - p_0 = p_1 - 0 = p_1 \Rightarrow p_2 = 2p_1$$
$$p_3 = 3p_1,\ p_i = i p_1$$
$$p_N = Np_1 = 1 \Rightarrow p_1 = 1/N \Rightarrow p_i = i/N$$