Combinations: order doesn't matter

Ex: Your elf-lord avatar can carry any 3 objects chosen from (1) sword, (2) knife, (3) staff, (4) ring, (5) laptop. How many combinations are there?

$$\frac{5!}{(5-3)!\,3!} = \frac{5\cdot 4}{2\cdot 1} = 10$$

More generally,

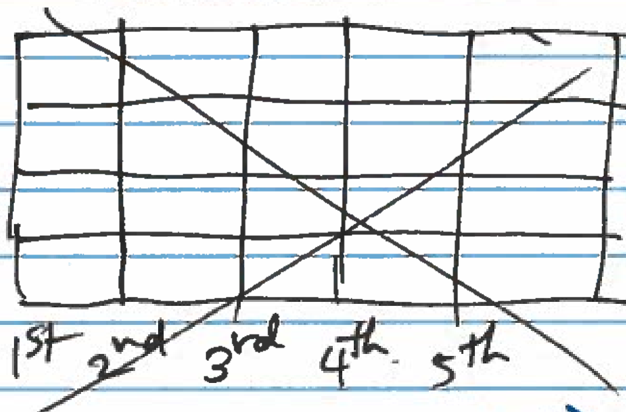
$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}$$

"n choose r", "binomial coefficients"

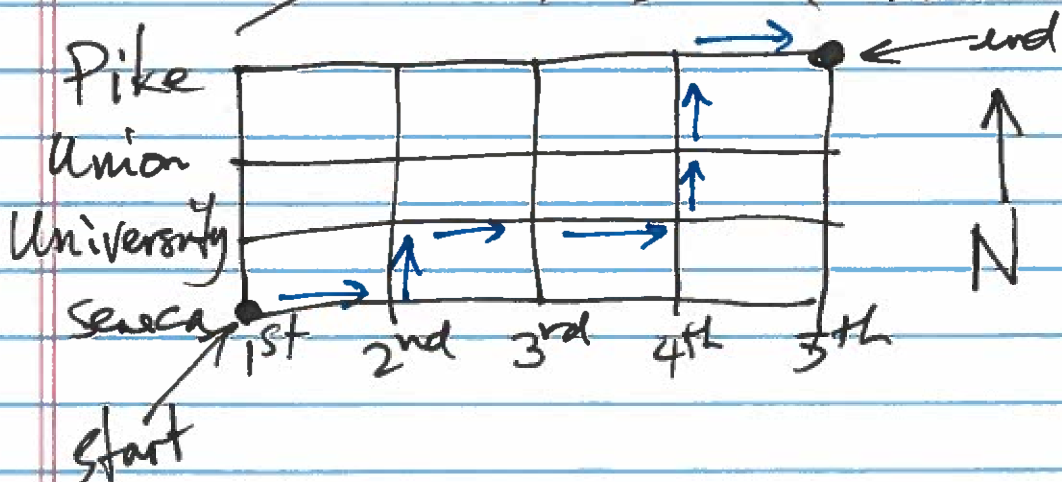
Ex: How many unordered pairs from n objects?

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2} \in \Theta(n^2)$$

Ex:



How many paths from Seneca & 1st to Pike & 5th, going N or E at each intersection?



Walk 7 blocks, 3 of them going N.

_ N _ _ N N _

$$\binom{7}{3} = \frac{7!}{3!\,4!} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 35$$

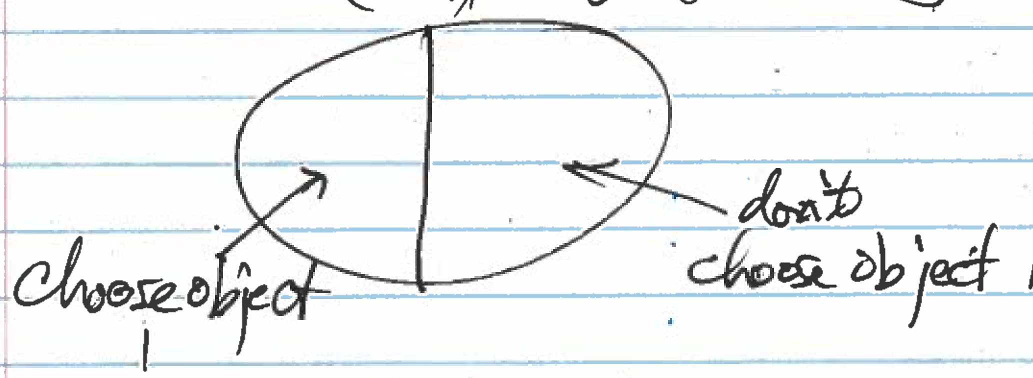
$$\binom{7}{4} = \frac{7!}{4!\,3!}$$

Identity: $\binom{n}{r} = \binom{n}{n-r}$, for any n, r.

$P(7,3)$

Identity: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, for $r \le n-1$.

Consider object 1. If it is chosen, then there are $\binom{n-1}{r-1}$ ways of choosing the remaining $r-1$. If it is not chosen, there there are $\binom{n-1}{r}$ ways of choosing all r.



<u>Binomial Theorem</u>: $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$

Proof by counting:

$(x+y)(x+y)(x+y)\cdots(x+y)$.

Pick either x or y from each factor, multiply those n variables, and then ~~colle~~ count the number of terms that are $x^i y^{n-i}$. How many are there? $\binom{n}{i}$, the number of ways in which I can choose i of the n factors to contribute an x to the product.

Ex: $(x+y)^3 = (x+y)(x+y)(x+y)$

$= xxx + xxy + xyx + yxx$

$+ xyy + yxy + yyx + yyy$

$= x^3 + 3x^2y + 3xy^2 + y^3$

<u>Corollary</u>: $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

<u>Proof</u>: $2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i}$

How many subsets does a set of size n have?