Instructions:

- **Answers:** When asked for a short answer (such as a single number), also show and explain your work briefly. If your answer is too large to evaluate easily, you can leave it in terms of factorials, combinations, etc., for instance $26^2$ or $26!/7!$ or $26(26\choose 7)$. Otherwise, do evaluate it completely, so that you get an idea of its magnitude. Your final answer should not have a summation in it.

- **Bundles:** The exercises in each homework assignment will be divided into 3 groups (to facilitate distribution to grading TAs). You will turn in 3 corresponding bundles. Write your full name in the upper left corner of each bundle’s top page, with your last name printed clearly in CAPITAL LETTERS. Each bundle should be stapled separately. We don’t supply the stapler.

This week’s turnin bundles: (A) exercises 1–3, (B) exercises 4–6, (C) exercises 7–9.

A1. Suppose you have a 20-card Schnapsen deck in a single stack. How many arrangements of the cards are there in the stack if . . .

(a) . . . only the rank matters? For example, swapping the positions of two jacks is considered the same arrangement.

(b) . . . only the suit matters? For example, swapping the positions of two clubs is considered the same arrangement.

A2. A Quidditch team (three Chasers, two Beaters, one Keeper, and one Seeker) consisting of 4 boys and 3 girls is to be chosen from a class of 12 boys and 10 girls. If you choose a particular 7 students, changing their team positions is considered to change the team. But a Chaser is a Chaser and a Beater is a Beater, so swapping the two Beaters does not change the team.

(a) How many different teams are possible if each child can play any position?

(b) How many different teams are possible, if 2 of the 12 boys only know how to be Seekers?

(c) How many different teams are possible, if 2 of the 10 girls only know how to be Seekers?

(d) How many different teams are possible, if one of the boys and one of the girls only know how to be Seekers?

A3. You sit down to a game of Schnapsen with the Maestro, who deals you the following hand:

$$
\spadesuit \_ \_ \\
\heartsuit \text{ATQ} \\
\spadesuit \text{AT} \\
\diamondsuit \_ \_ \\
$$

The face-up trump showing on the table is $\heartsuit J$ and you are on lead at the very first trick. With this beautiful hand, you decide to close the stock immediately. You then lead $\heartsuit A$, followed by your other 4 cards in a random order.

(a) Are you guaranteed to take all the tricks?

(b) What is the maximum number of trick points the Maestro could be holding in his starting hand, yet you would lose the deal?

(c) What is the total number of possible starting hands for the Maestro in this deal?
(d) How many of the Maestro’s possible starting hands will cause you to lose this deal? (Do not enumerate them. Instead, use counting ideas in order to arrive at a simple way to solve this.)

(e) What is the probability that you will lose this deal?

B4. A piano octave consists of 12 notes in ascending order. Five of them are black key notes and seven are white key notes. A composer by the name of A. Tonal is experimenting with the idea that a melody must be 7 notes from this single octave, 3 of them black and 4 of them white. The notes of a melody need not be distinct: you can use the same note 2 or more times. How many possible melodies are there if:

(a) there are no further restrictions?
(b) the black and white notes must alternate?
(c) the 3 black notes cannot all be adjacent in the melody?
(d) no note is allowed to be repeated?
(e) no note is allowed to be repeated and the black notes must be either ascending or descending in the melody?

B5. How many ways are there to pair up 102 students into 51 pairs? The order of the pairs does not matter.

B6. Let \( f(n) = \sum_{i=0}^{n} (-3)^i \binom{n}{i} \). Prove that \( f(n) = 2^n \) for all positive even integers \( n \), and \( f(n) = -2^n \) for all positive odd integers \( n \).

C7. Suppose we have a one-dimensional integer array \( A \) of size \( n \), where \( n \geq 4 \). We insert the integers 0, 1, 2, …, \( n-1 \) into this array, with each integer appearing exactly once in the array. We will say that \( i \) is at its own index if \( A[i] = i \). How many arrangements are there in which:

(a) at least 1 entry is not at its own index?
(b) at least 2 entries are not at their own indices?
(c) at least 3 entries are not at their own indices?
(d) at least 4 entries are not at their own indices?

C8. Consider the following equation: \( a_1 + a_2 + a_3 + a_4 + a_5 = 80 \). A solution to this equation over the nonnegative integers is a choice of a nonnegative integer for each of the variables \( a_1, a_2, a_3, a_4, a_5 \) that satisfies the equation. For example, \( a_1 = 15, a_2 = 3, a_3 = 15, a_4 = 0, a_5 = 47 \) is a solution. To be different, two solutions have to differ on the value assigned to some \( a_i \). How many different solutions are there to the equation? (Hint: Think about splitting a sequence of 80 1’s into 5 blocks, each block consisting of consecutive 1’s in the sequence. The number of 1’s in the \( i \)-th block corresponds to the value of \( a_i \). Note that the \( i \)-th block is allowed to be empty, corresponding to \( a_i = 0 \).)

C9. You have 8 quarters left until graduation, and you want to take each of the following 8 courses, one per quarter: CSE 333, CSE 351, CSE 421, CSE 427, CSE 428, CSE 446, CSE 451, and CSE 452. Each course is offered every quarter, but you must obey the following prerequisite requirements:

- CSE 351 must be taken before CSE 333.
- CSE 333 must be taken before CSE 451.
- CSE 451 must be taken before CSE 452.
- CSE 427 must be taken before CSE 428.

(a) How many possible schedules are there?
(b) You are convinced that you will fail CSE 452 unless you take it the quarter immediately after you take CSE 451. How many possible schedule are there that satisfy this additional constraint? (No classes are taught in the summer, and autumn is considered to be immediately after spring.)
Extra Credit Problem:

Instructions: Extra credit problems will be worth very few points; this one is worth only 2 points, compared to the total of 73 for the rest of Assignment #1. Just work on it if you enjoy an extra challenge. It will also be graded all-or-nothing: if you get the correct final answer and support it with a correct derivation, you will get the full 2 points, otherwise 0. If you do this one, include it at the end of Bundle A, clearly marked “Extra Credit”.

You must have seen requirements for passwords many times. For instance, here are the requirements from [https://technet.microsoft.com/en-us/library/cc756109](https://technet.microsoft.com/en-us/library/cc756109):

1. Has at least 7 and no more than 127 characters.
2. Does not contain the user’s username, real name, or company name as a substring.
3. Does not contain a complete dictionary word.
4. Contains characters from each of the following four groups:
   (a) Uppercase letters
   (b) Lowercase letters
   (c) Numerals
   (d) Other symbols found on the keyboard

If you were concerned with security, you might want to count the number of possible passwords, in order to make sure that a cyberbot could not try them all. We are going to simplify the password requirements above quite a bit, and your task is to count the number of possible passwords exactly.

For the purposes of this problem, then, a username consists of exactly 5 lower case letters, with no repeated letter. A password must satisfy the following requirements:

1. Has at least 7 and no more than 14 characters.
2. Does not contain the user’s username as a substring.
3. Contains characters from each of the following two groups:
   (a) Lowercase letters
   (b) Numerals

(a) For a given user, how many possible passwords are there? You must give an exact numerical answer and must support your answer by showing how you derived it. You are welcome to write a small script or program to help you evaluate your final formula, which may not lend itself to a simple closed form. Be careful that the language you choose supports arbitrary precision, because your final answer will have more than 20 digits.

(b) Why did we insist that usernames contain no repeated letter?