CSE 312: Foundations of Computing II
Quiz Section #1: Permutations, Combinations, and the Pigeonhole Principle

Review/Mini-Lecture/Main Theorems and Concepts From Lecture

Product Rule: Suppose there are \( m_1 \) possible outcomes for event \( A_1 \), then \( m_2 \) possible outcomes for event \( A_2 \), ..., \( m_n \) possible outcomes for event \( A_n \). Then there are \( m_1 \cdot m_2 \cdot ... \cdot m_n = \prod_{i=1}^{n} m_i \) possible outcomes overall.

Number of ways to order \( n \) distinct objects: \( n! = n(n-1) \cdots (3)(2)(1) \).

Permutations (number of ways to “pick” \( k \) objects out of \( n \) when order matters): \( \frac{n!}{(n-k)!} = nP_k = P(n,k) \).

Combinations (number of ways to “choose” \( k \) objects out of \( n \) when order doesn’t matter): \( \binom{n}{k} = \frac{n!}{(n-k)!k!} = C(n,k) \).

Binomial Theorem: \( \forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}, (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \).

Principle of Inclusion-Exclusion (PIE):
2 events: \( |A \cup B| = |A| + |B| - |A \cap B| \)
3 events: \( |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \)
In general: +singles – doubles + triples – quads + …

Pigeonhole Principle: If there are \( n \) pigeons with \( k \) holes and \( n > k \), then at least one hole contains at least \( \lceil n/k \rceil \) pigeons.

Complementary Counting (Complementing): If asked to find the number of ways to do X, you can find the total number of ways and then subtract the number of ways to not do X.

Multinomial Coefficients: Suppose there are \( n \) objects, but only \( k \) are distinct, with \( k \leq n \). (For example, “godoggy” has \( n = 7 \) objects [characters] but \( k = 4 \) distinct objects, \{g, o, d, y\}). Let \( n_i \) for \( i = 1, ..., k \) be the number of times object \( i \) appears. (For example, \{3,2,1,1\} continuing the “godoggy” example). The number of ways to arrange the \( n \) objects is

\[
\frac{n!}{n_1!n_2!...n_k!} = \binom{n}{n_1, n_2, ..., n_k}
\]


**Exercises**

1. In the game of bridge, a hand consists of 13 cards. Given a bridge hand consisting of 5 spades, 2 hearts, 3 diamonds, and 3 clubs, in how many ways can the hand be arranged so that the cards of each suit are together…

   a. … but not necessarily sorted by rank within each suit?

\[
\frac{4! \cdot 5! \cdot 2! \cdot 3! \cdot 3!}{4! \cdot \text{ways to order all the suits,} \quad 5! \cdot \text{ways to order the spades} \\
2! \cdot \text{ways to order the hearts,} \quad 3! \cdot \text{ways to order the diamonds} \\
3! \cdot \text{ways to order the clubs}
\]

b. … and each suit is sorted in ascending rank order?

\[
4! \\
\text{4! \rightarrow ways to order the suits.} \\
\text{There's only 1 way to order each of the cards within the suits}
\]

c. … and each suit is sorted in ascending rank order and the suits are arranged so that the suit colours alternate?

\[
4 \times 2 \times 1 \times 1 \\
\text{4 options for what suit is first} \\
\text{2 options for the next suit because it has to be of the other colour} \\
\text{then one option each for the remaining two suits}
\]

2. How many ways are there to select 5 cards from a standard deck of 52 cards, where the 5 cards contain cards from at most two suits if:

   a) order does not matter

   Case 1: all from the same suit – choose 1 of 4 suits, and 5 cards from that suit

\[
\binom{4}{1} \binom{13}{5}
\]

   Case 2: from two suits – choose 2 of 4 suits: 1 from the first and 4 from the second, 2 from the first and 3 from the second, etc.)

\[
\binom{4}{2} \left[ \binom{13}{1} \binom{13}{4} + \binom{13}{2} \binom{13}{3} + \binom{13}{3} \binom{13}{2} + \binom{13}{4} \binom{13}{1} \right]
\]

   Our total is

\[
\binom{4}{1} \binom{13}{5} + \binom{4}{2} \left[ \binom{13}{1} \binom{13}{4} + \binom{13}{2} \binom{13}{3} + \binom{13}{3} \binom{13}{2} + \binom{13}{4} \binom{13}{1} \right]
\]
Let’s talk about an incorrect solution

Step 1: First choose the two suits from which the cards will come: \( \binom{4}{2} \) possibilities

Step 2: Then choose the 5 cards from among the 26 possible cards of those suits: \( \binom{26}{5} \)

Thus, the total number is \( \binom{4}{2} \binom{26}{5} \) ways is \( \binom{4}{2} \binom{26}{5} \)

NOT!

The problem is that this method overcounts some hands. In particular, a hand consisting of cards that are entirely from one suit, say hearts, will be counted 3 times:

Once when

- the two suits selected are Hearts/Spades, once when
- the two suits selected are Hearts/Diamonds and once when
- the two suits selected are Hearts/Clubs.

Applying "The Sleuth Principle" from lecture on 4/1, given an outcome selected according to some application of the product rule, we need to be able to reconstruct exactly what choice was made at each step, or else we have made a mistake.

When we see an outcome consisting of all hearts, we cannot reconstruct the choice made in the first step -- it could have been any of the 3 possibilities mentioned above.

To correct this, one can subtract off the overcounted stuff which is

\[ 2 \binom{4}{1} \binom{13}{5} \]

b) order matters

Just 5! Times the previous answer, since we can permute the 5 distinct cards that many ways.

\[ 5! \left( \binom{4}{1} \binom{13}{5} + \binom{4}{2} \left[ \binom{13}{1} \binom{13}{4} + \binom{13}{2} \binom{13}{3} + \binom{13}{3} \binom{13}{2} + \binom{13}{4} \binom{13}{1} \right] \right) \]

3. At a dinner party, all of the people present are to be seated at a circular table. Suppose there is a nametag at each place at the table and that nobody sits down in their correct place. Show that it is possible to rotate the table so that at least two people are sitting in the correct place.

Let \( r_i \), \( 1 \leq i \leq n \) be the offset of \( i \) from his/her correct seat, i.e., the number of positions by which the table would need to be rotated in the clockwise direction to bring the \( i^{th} \) nametag to the corresponding person. These offsets range from 1 to \( n - 1 \), since \( 0/n \) means that the person is seated correctly, which by assumption cannot be true. Therefore, each offset has \( n - 1 \) possible different values. There are \( n \) such offsets (one for each person), that can take \( n - 1 \) possible different values. By the pigeonhole principle, there exists an \( i \) and \( j \), where \( i \neq j \), such that \( r_i = r_j \). If you rotate the table by \( r_i \), \( i \) and \( j \) are seated
correctly.

Let the \( n \) guests be pigeons and the \( n \) possible rotations/solutions be the pigeonholes. There are no pigeons in the first hole because none of the guests are seated correctly. This leaves \( n - 1 \) pigeonholes and \( n \) pigeons, so by the pigeonhole principle, two of them must go in the same hole. That is, there must be some rotation where at least two of the guests are sitting in the correct positions.

4. Consider a set of 25 people that form a social network. (The structure of the social network is determined by which pairs of people in the group are “friends”.) How many possibilities are there for the structure of this social network?

\[
\binom{25}{2}
\]
possible undirected edges, each is either there or not \( \to 2^{\binom{25}{2}} \).

5. You want to choose a team of \( m \) people for your startup company from a pool of \( n \) applicants, and from these \( m \) people you want to choose \( k \) to be the team managers. You took CSE 312, so you know you can do this in \( \binom{n}{m} \binom{m}{k} \) ways. But your CFO comes up with the formula \( \binom{n}{k} \binom{n-k}{m-k} \). Before dumping on your CFO, you decide to check his answer against yours.

Give a combinatorial proof that your CFO’s formula agrees with yours.

Let’s break down both solutions and see why they’re equivalent.

In our solution, we are choosing our team \( \binom{m}{m} \) first from the applicants \( \binom{n}{n} \) – that is represented by the term \( \binom{n}{m} \). For each team that we have, we now have to choose our managers \( \binom{k}{k} \) using the product rule, we multiply the number of teams we have with the number of ways we can choose managers from the team. This is represented by the term \( \binom{m}{k} \).

The CFO decides that he wants to choose his managers from the applicant pool first. He does this with the term \( \binom{n}{k} \). Now we have \( n - k \) people in the pool because we’ve removed \( k \) people from it. And our team comprises of \( m \) people, and since we already have \( k \) people, we need to pick another \( m - k \). This is represented by the term \( \binom{n-k}{m-k} \).

In the end, we end up counting the same number of teams, but in different ways.

6. Suppose we have 3 diamonds and 3 hearts from a standard deck. How many ways are there to arrange the cards if they have to alternate suit?

Method 1: 6 possible cards for the first location, then 3 because you can’t choose the same suit. Then 2 for the third location, because the suit is determined by the first location and there are only 2 cards left in that suit. Similarly 2 for the fourth location. Then 1 choice for each of the fifth and sixth locations.

\[
6 \times 3 \times 2 \times 2 \times 1 \times 1
\]

Method 2: Find the arrangements individually for each of the suits: 3! for each suit. They have to be alternating, but there are 2 choices for which suit comes first, and then the order will be determined.

\[
2 \times 3!^2
\]
Check that the answers are equivalent.

7. There are 80 restaurants on the Ave. 36 serve wine, 30 serve pizza, 32 serve hamburgers, 13 serve wine and pizza, 19 serve pizza and hamburgers, 8 serve wine and hamburgers, and 6 serve all three.

Make a Venn diagram, then the answers are trivial.

![Venn Diagram]

a) How many restaurants serve none of the three? 16
b) How many restaurants can minors dine at (no wine)? 4+13+11+16=44
c) How many restaurants only serve pizza? 4

8. How many ways are there to choose three initials that have two being the same or all three being the same?

Complementary counting. Count the total $26^3$ and subtract the number with all distinct initials $26 \times 25 \times 24$ to get $26^3 - 26 \times 25 \times 24$.

9. There are 40 seats and 40 students in a classroom. There are 10 seats in the front row, and 6 students are near-sighted so they must sit in the front row. How many seating arrangements are possible?

Pick the 6 for the front since placement matters, to get $\binom{10}{6}$, then the other 34 can be arranged in any order $34!$, so we get $\binom{10}{6} \times 34!$.

10. A chef is preparing desserts for each day of the week (Su, M, ... , F, S). Each day, he prepares one of: apple pie, cake, ice cream sundae, or fruit platter. Since it is someone’s birthday on Thursday, he must prepare cake. However, he cannot serve the same dessert on any two consecutive days. How many dessert combinations are possible?

Start with Thursday – there is one choice. Then for each of the rest of the days, there are exactly 3 choices. So the answer is $3^6$.

11. Rabbits Peter and Pauline have three offspring: Flopsie, Mopsie, and Cotton-tail. These rabbits are to be distributed to 4 different pet stores such that no store gets both a parent and a child. Not every store must receive a rabbit. How many ways can this be done?

Case 1: Both parents go to the same store.
There are \( \binom{4}{1} \) ways to choose which store. Then each of the 3 baby rabbits has 3 choices for which store. So we get \( \binom{4}{1}3^3 = 108 \).

Case 2: The parents go to a different store.

There are 4 stores the first Peter can go to, then 3 stores Pauline can go to. Then each of the 3 baby rabbits has 2 choices for which store. So we get \( 4 \times 3 \times 2^3 = 96 \).

Adding these (since they are mutually exclusive), we get \( 108 + 96 = 204 \).

12. There are 6 women and 7 men in a ballroom dancing class. If 4 men and 4 women are chosen and paired off, how many pairings are possible?

First choose 4 men and 4 women, \( \binom{7}{4} \) and \( \binom{6}{4} \) respectively. Then, fix the order of men \( M_1, M_2, M_3, M_4 \).

You can see that there are \( 4! \) ways to assign the women to each man, so we have a total of \( \binom{7}{4}\binom{6}{4}4! \) ways.

13. How many ways are there to seat all of the TA’s (9 of us) and Professor Karlin in a row if Professor Karlin cannot sit on either end?

Directly: She must sit in one of the eight middle seats, and then the rest of us can sit anywhere, so there are \( 8 \times 9! \) ways.

Complementing: The total number of ways is \( 10! \). The number of ways if she sits on either end is \( 2 \times 9! \) Since she has two choices for which end, then the rest of us can sit anywhere.

Then we have to subtract to get \( 10! - 2 \times 9! = 10 \times 9! - 2 \times 9! = 8 \times 9! \).

14. For each part, find the minimum number of cards that have to be dealt from a standard 52-card deck to guarantee a:

   a) Single Pair \( 14 \)
   b) Two Distinct Pairs \( 17 \)
   c) Full House [3 of a kind, and 2 of a kind] \( 27 \)
   d) Straight [5 in a row: lowest is A2345, highest is 10JQKA] \( 45 \)
   e) Flush [all 5 cards same suit] \( 17 \)
   f) Straight Flush [requirements of Straight and Flush] \( 45 \)

15. For each part, find the number of ways to get the following when dealt exactly 5 cards from a standard 52-card deck.

   a) Single Pair (5 cards AABCD) \( \binom{13}{1}\binom{4}{2}\binom{12}{3}4^3 \)
   b) Two Distinct Pairs (5 cards AABBC)
\[
\binom{13}{2} \binom{4}{2}^2 44
\]

c) Full House [3 of a kind, and 2 of a kind] (5 cards AAABB)

\[
\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2}
\]

d) Straight [5 in a row: lowest is A2345, highest is 10JQKA]

\[
10 \times 4^5
\]

e) Flush [all 5 cards same suit]

\[
\binom{4}{1} \binom{13}{5}
\]

f) Straight Flush [requirements of Straight and Flush]

\[
10 \times 4
\]

16. How many ways are there to arrange 8 people in a circle?

\[
\frac{8!}{8} = 7!, \text{ since there are 8 placements for the “first” person, and can be rotated 8 ways.}
\]