CSE 312 Homework 2, Due Wednesday, April 13, before class

Instructions:
• When asked for a short answer (such as a single number), also show and explain your work briefly. Unless you are asked to, leave your answer in terms of factorials, combinations, etc., for instance \(26^7\) or \(26! / 7!\) or \(26 \cdot \binom{26}{7}\).
• Please write down on your homework the names of all people you discussed the homework with.
• Turn-in procedure:
  – You will turn in your homework on paper, in 3 separate bundles - you may hand-write your homework legibly, or type it and print it out. If we can’t read your homework, we won’t grade it.
  – Each bundle needs to have your name at the top, and the bundle letter A, B, or C.
  – You need to staple the papers in each bundle - homework that is not stapled is not accepted - turn in your favorite sheet for that bundle if you have multiple unstapled sheets.

This week’s bundles: (A) problems 1-2; (B) problems 3-6; (C) problems 7-8

Problems

A.1 For each of the following scenarios first answer the following two questions and then answer the question stated. (a) What is the sample space and how big is it? (b) What is the probability of each outcome in the sample space?

(a) You flip a fair coin 100 times. What is the probability that all 100 tosses are the same? What is the probability of exactly 50 heads?
(b) You roll 2 fair dice. What is the probability that the sum of the two values showing is 5?
(c) You are given a random 5 card poker hand (selected from a single deck). What is the probability you have a flush (all cards have the same suit)?
(d) 20 labeled balls are placed into 10 labeled bins (with each placement equally likely). What is the probability that bin 1 contains at least one ball?
(e) There are 30 psychiatrists and 24 psychologists attending a certain conference. Three of these 54 people are randomly chosen to take part in a panel discussion. What is the probability that at least one psychologist is chosen? What is the probability that exactly three psychologists are chosen?
(f) You buy a dozen bagels choosing from 3 different varieties (plain, garlic and pumpernickel). Bagels of the same type are indistinguishable. What is the probability that you have at least 2 of each type?

A.2 Consider the question: Suppose a team of 10 players is selected at random out of eighteen possible people, where 15 are women and 3 are men. Each team (unordered) is equally likely.

Each of the \(\binom{18}{10}\) teams is equally likely, so that is the size of the sample space. Let \(E\) be the event that the team selected has at least one man. Then

\[
Pr(E) = \frac{|E|}{\binom{18}{10}}
\]

To count \(|E|\), first pick one man out of the 3 possible, and then pick the rest of the team from the remaining people. So

\[
|E| = \binom{3}{1} \cdot \binom{17}{9} \quad \text{and hence} \quad Pr(E) = \frac{\binom{3}{1} \cdot \binom{17}{9}}{\binom{18}{10}}
\]
Explain what is wrong with this solution. If there is over-counting in the calculation of $|E|$, characterize all teams that are counted more than once, and how many times each such team is counted using the above solution.

What is the correct answer for $Pr(E)$?

B.3 Probability can be strange. Suppose you have 3 dice: Die A has 3 sides with the numbers 2, 4 and 9 on it. Die B has 3 sides with the numbers 1, 6 and 8. Die C has 3 sides with the numbers 3, 5, and 7 on it. (Let's not worry about any physical impossibilities here.) You toss all 3 dice and each outcome is equally likely. What is the probability that die A shows a larger number than die B? What is the probability that die B shows a larger number than die C? What is the probability that die C shows a larger number than die A?

B.4 Consider two probability spaces: In the first, the outcomes are all unordered subsets of size $k$ out of a set of $n$ distinct elements. In the second, the outcomes are all ordered subsets of size $k$ out of a set of $n$ distinct elements. Both probability distributions are uniform.

Let $E$ be any event in the first probability space. Let $F$ be the event in the second probability space defined as follows:

$$F = \{ \text{All ordered sequences } (i_1, i_2, \ldots, i_k) \text{ such that the unordered subset } \{i_1, \ldots, i_k\} \text{ is an element of } E \}.$$ 

Show that the probability of $E$ in the first probability space is equal to the probability of $F$ in the second probability space. (An example of this type of scenario is the ding dongs/twinkies problem we did in class.)

B.5 Consider a weighted die such that

- $Pr(1) = Pr(2)$,
- $Pr(3) = Pr(4) = Pr(5) = Pr(6)$, and
- $Pr(1) = 2Pr(3)$.

What is the probability that the outcome is 5 or 6?

B.6 Three people are brought into a room. A hat is placed on each person’s head. The hat is equally likely to be Red or Blue. (So each of the 8 possibilities is equally likely.) Each person sees the colors of the other people’s hats, but not their own. Each person, without communication, writes down one of the following: "My hat is red", "My hat is blue" or "Pass". All three people will be shot unless (a) at least one of them doesn’t pass, and (b) everyone who doesn’t pass is right about his/her own hat color. Importantly, they can agree ahead of time on a strategy, with the hopes of surviving.

- What is the probability that they are not all shot if each person guesses the color of his/her own hat?
- What is the probability that they are not all shot if two of them pass and 1 of them guesses?
- What is the probability that they are not all shot if they use the following strategy: Each person looks at the other two hats. If they are both blue, then the person guesses red. If they are both red, then the person guesses blue. If they are different, the person passes.

C.7 You are shown two envelopes and told the following facts:

- Each envelope has some number of dollars in it, but you don’t know how many.
- The amount in the first envelope is different from the amount in the second.
- Although you don’t know exactly how much money is in each envelope, you are told that it is an integer number of dollars that is at least 1 and at most 100.
• You are told that you can pick an envelope, look inside, and then you will be given a one-time option to switch envelopes (without looking inside the new envelope). You will then be allowed to keep the money in envelope you end up with.

Your strategy is the following:

(a) You pick an envelope uniformly at random.
(b) You open it and count the amount of money inside. Say the result is $x$.
(c) You then select an integer $y$ between 1 and 100 uniformly at random.
(d) If $y > x$, you switch envelopes, otherwise you stay with the envelope you picked in step (a)

Show that you have a better than 50-50 chance of taking home the envelope with the larger amount of money in it. More specifically, suppose the two envelopes have $i$ and $j$ dollars in them respectively, where $i < j$. Calculate the probability that you take home the envelope with the larger amount of money.

C.8 Consider the following variant of the Monty Hall Problem: There are 5 doors and the host promises to open 2 of the doors with goats behind them after the contestant has chosen a door. What is the probability that the contestant wins a car by switching? State all your assumptions.