Random variables

Discrete random variable: take a finite or countable number of values

Two Coins Tossed

X: \{TT, TH, HT, HH\} \rightarrow \{0, 1, 2\} counts the number of heads

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Definition: Expectation

The expectation, or expected value of a random variable X is written as E[X], and is

\[ E[X] = \sum_{t \in S} X(t) \Pr(t) = \sum_k k \Pr[X = k] \]

A Quick Calculation...

What if I flip a coin 3 times? What is the expected number of heads?

E[X] = (1/8)\times 0 + (3/8)\times 1 + (3/8)\times 2 + (1/8)\times 3 = 1.5

But \Pr[ X = 1.5 ] = 0

Moral: don’t always expect the expected. \Pr[ X = E[X] ] may be 0!
Functions of a R.V.

Suppose that $X$ is a random variable defined on probability space $(S, p(.))$.
Then $Y = g(X)$ is also a random variable on the same probability space (if defined everywhere).

E.g., $Y = X^2$
$Z = 3^X + 2$

Expectation of a function of a r.v.

$$E(X) = \sum_k kPr(X = k) = \sum_{\omega \in S} X(\omega)Pr(\omega)$$
$$E(g(X)) = \sum_k g(k)Pr(X = k) \quad k \text{ in range of } X$$

Operations on R.V.s

You can define any random variable you want on a probability space.
Given a collection of random variables, you can sum them, take differences, or do most other math operations…

E.g., $(X + Y)(t) = X(t) + Y(t)$
$(X*Y)(t) = X(t) * Y(t)$
$(X^Y)(t) = X(t)^{Y(t)}$

Random variables and expectations allow us to give elegant solutions to problems that seem really really messy…

If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?

On average, in class of size $m$, how many pairs of people will have the same birthday?

Pretty messy with direct counting…

The new tool is called “Linearity of Expectation”
Linearity of Expectation

If \( Z = X + Y \), then
\[
E[Z] = E[X] + E[Y]
\]
Without any conditions on \( X \) and \( Y \)

By Induction

\[
E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]
\]
The expectation of the sum = The sum of the expectations

Let’s see why Linearity of Expectation is so useful...

If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?

Hmm…

\[
\sum_k k \cdot Pr(\text{exactly } k \text{ letters end up in correct envelopes})
\]
\[
= \sum_k k \cdot (\ldots\text{aargh}!!\ldots)
\]

If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?

Indicator Random Variables

For any event \( A \), can define the “indicator random variable” for event \( A \):

\[
X_A(t) = \begin{cases} 
1 & \text{if } t \in A \\
0 & \text{if } t \notin A 
\end{cases}
\]

\[
E[X_A] = 1 \times Pr(X_A = 1) = Pr(A)
\]
Use Linearity of Expectation
Let \( A_i \) be the event the \( i \)th letter ends up in its correct envelope
Let \( X_i \) be the “indicator” R.V. for \( A_i \)
Let \( Z = X_1 + \ldots + X_{100} \)
We are asking for \( E[Z] \)

So, in expectation, 1 letter will be in the same correct envelope
Pretty neat: it doesn’t depend on how many letters!

Ex. #2
We flip \( n \) coins of bias \( p \). What is the expected number of heads?
We could do this by summing
\[
\sum_k k \Pr(X = k) = \sum_k k \binom{n}{k} p^k (1-p)^{n-k}
\]
But now we know a better way!

Use Linearity of Expectation
Let \( A_i \) be the event the \( i \)th letter ends up in its correct envelope
Let \( X_i \) be the “indicator” R.V. for \( A_i \)
Let \( Z = X_1 + \ldots + X_{100} \)
We are asking for \( E[Z] \)

\[
E[X_i] = \Pr(A_i) = 1/100
\]
So \( E[Z] = 1 \)

Use Linearity of Expectation
General approach:
View thing you care about as expected value of some R.V
Write this R.V as sum of simpler R.Vs
Solve for their expectations and add them up!

Linearity of Expectation!
Let \( X \) = number of heads when \( n \) independent coins of bias \( p \) are flipped
Break \( X \) into \( n \) simpler R.Vs:
\[
X_i = \begin{cases} 
1 & \text{if the } i \text{th coin is heads} \\
0 & \text{if the } i \text{th coin is tails}
\end{cases}
\]
\[
E[X] = E[\sum_i X_i] = \sum_i E[X_i] = \sum_i p = np
\]
On average, in class of size m, how many pairs of people will have the same birthday?

$$\sum_k k \Pr(\text{exactly } k \text{ pairs}) = \sum_k k \left(\ldots\text{aargh!!!!}\ldots\right)$$

Use linearity of expectation

Suppose we have m people each with a uniformly chosen birthday from 1 to 365

$X = \text{number of pairs of people with the same birthday}$

$E[X] = ?$

$X = \text{number of pairs of people with the same birthday}$

$X_{jk} = 1$ if person j and person k have the same birthday; else 0

$E[X_{jk}] = \left(\frac{1}{365}\right) 1 + \left(1 - \frac{1}{365}\right) 0 = \frac{1}{365}$

$E[X] = E[\sum_{1 \leq j < k \leq m} X_{jk}]$

$= \sum_{1 \leq j < k \leq m} E[X_{jk}]$

$= m(m-1)/2 \times \frac{1}{365}$

$\therefore \quad \frac{m^2}{2} \div 365$

Type Checking

$P( B )$  \quad B must be an event

$E( X )$  \quad X must be a R.V.

cannot do $P(\text{R.V.})$ or $E(\text{event})$