Continuous Random Variables

Seen discrete r.v.'s

\((\Omega, \mathcal{F}, \mathbb{P})\) prob space

\(X : \Omega \rightarrow \mathbb{R}\)

\(\mathbb{P}\) m.f. \(\mathbb{P}_X(x_i) = \mathbb{P}(X = x_i)\)

C.D.F. \(F_X(x) = \mathbb{P}(X \leq x)\)

\(F_X(x) = \sum_{x_i \leq x} \mathbb{P}_X(x_i)\)

\(p_X(x_i) = F_X(x_i) - F_X(x_{i-1})\)

Suppose we want to represent continuous valued random var

e.g. draw a random \(x \in (0, 1]\)

Discrete approx: \(\mathcal{X} = \{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\}\)

\(p_X(x) = \begin{cases} \frac{1}{n} & x = \frac{i}{n} \\ 0 & \text{otherwise} \end{cases} \quad i = 1 \ldots n\)

\(F_X(x) = \begin{cases} \frac{j}{n} & \frac{j}{n} \leq x < \frac{j+1}{n}, \quad j = \ldots n-1 \end{cases} \)
\[ F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 1 & x > 1 \end{cases} \]

\[ \lim_{n \to \infty} P_X(x) = 0 \]

The notion of probability mass function does not make sense anymore.

**Instead**

**Probability density function**

\[ f_X(x) = \frac{d}{dx} F_X(x) \]

\[ F_X(x) = \int_{-\infty}^{x} f_X(x) \, dx \]

Recall \( F_X(x) = \sum_{y \leq x} p_X(y) \)

**Properties:**

- \( F_X(x) \) monotone \( \uparrow \) from 0 to 1
- \( \int_{-\infty}^{\infty} f_X(x) \, dx = 1 \)
- \( f_X(x) \geq 0 \) (but not necessarily \( \leq 1 \))
- \( \Pr(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx \)
Continuous setting

\[ E(X) = \int_{-\infty}^{\infty} x f(x) \, dx \]

\[ E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) \, dx \]

Discrete setting

\[ E(X) = \sum_{x \in \text{Range}(X)} x \cdot p(x) \]

\[ E(X) = \sum_{x \in \text{Range}(X)} g(x) \cdot p(x) \]

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Example: X cont. r.v.

\[ f(x) = \begin{cases} C(4x-2x^3) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases} \]

1) What is \( C \)?

\[ \int_{0}^{2} C(4x-2x^3) \, dx = 1 \]

\[ \Rightarrow C = \frac{2}{8} \]

2) What is \( \Pr(X > 1) \)?

\[ \int_{1}^{2} \frac{2}{8}(4x-2x^3) \, dx \]
3) What is $E(X)$?

$$E(X) = \frac{3}{8} \int_{0}^{2} (4x-x^2) \, dx$$

**Application:** Auctions

1st price auction: winner pays his bid

2nd price auction: winner pays 2nd highest bid
How does bidder decide how to bid?

Asks: how happy do I expect to be with this bid?

$$\text{happiness} = \begin{cases} \text{value} - \text{price} & \text{if win} \\ 0 & \text{otherwise} \end{cases}$$

Bids "best response" to how other guy bids.

Ex: $v_1 = 80$ \hspace{1cm} $b_2 = 60$

**1st price auction analysis**

Suppose I promise you the other guy will bid $B_2 = \frac{v_2}{2}$

$v_2 \sim U[0, 100]$

How should you bid when your value is $v$.

Choose bid $b$ to maximize expected payoff.

$$(v - b) \Pr(\text{win}) = (v - b) \Pr(b > \frac{v_1}{2})$$

$$= (v - b) \Pr(v_1 < 2b)$$

$$= (v - b) \frac{2b}{100}$$
\[
\max_b \frac{(v-b)b}{100} = \max_b (v-b)b
\]

Take derivative:
\[-b + (v-b) = 0 \implies b = \frac{v}{2}
\]

bidding \( \frac{v}{2} \) is an equilibrium.

Each bidder is playing a best response to bid of other.

\[
E(\text{auctioneer's revenue}) = E\left[ \max\left( \frac{V_1}{100}, \frac{V_2}{100} \right) \right]
\]

\[
F_X(x) = \Pr \left[ \max\left( \frac{V_1}{100}, \frac{V_2}{100} \right) \leq x \right]
\]

\[
= \Pr \left( V_1 \leq 2x, V_2 \leq 2x \right)
\]

\[
= \frac{2x}{100} \cdot \frac{2x}{100}
\]

\[
= \frac{(2x)^2}{100^2}
\]

\[
\Rightarrow f_X(x) = \frac{8}{100^2} x
\]

\[
= \frac{8}{100^2} \int_0^x x^2 dx = \frac{8}{100^2} \left[ \frac{x^3}{3} \right]_0^{50}
\]

\[
= \frac{100}{3}
\]
2nd price auction: "dominant strategy" to report truthfully

\[ E[\text{auctioneer revenue}] = E[\min (V, V_{0})] \]

\[ 1 - F_{Y}(y) = \Pr(\min (V, V_{0}) \geq y) \]

\[ = \Pr(V, Y \geq y, V_{0} \geq y) \]

\[ = (1 - \frac{x}{100})(1 - \frac{y}{100}) \]

\[ \Rightarrow F_{Y}(y) = 1 - (1 - \frac{y}{100})^{2} \]

\[ f_{Y}(y) = \frac{d}{dy} F_{Y}(y) = \frac{2}{100} (1 - \frac{y}{100}) \]

\[ E[\min (V, V_{0})] = \int_{0}^{100} y \cdot \frac{2}{100} (1 - \frac{y}{100}) \, dy \]

\[ = \frac{2}{100} \left[ \frac{y^{2}}{2} - \frac{y^{3}}{300} \right]_{0}^{100} \]

\[ = \frac{100}{3} \quad \text{same as 1st price!!} \]