CSE 312 Foundations II

Counting

from slides by W.L. Ruzzo and others

the basic principle of counting: the product rule

If there are
   n outcomes/choices for some event A,
   sequentially followed by m outcomes/choices for
   event B,
then there are \( n \cdot m \) outcomes/choices overall.

\[ A, n = 4 \]
\[ B, m = 2 \]
\[ 4 \times 2 = 8 \text{ outcomes} \]

How many ways are there to do \( X \)?

E.g., \( X = \) “choose an integer 1, 2, ..., 10”

E.g., \( X = \) “Walk from 1st & Marion to 5th & Pine, going only North or East at each intersection.”

The Point:
Counting gets hard when numbers are large, implicit and/or constraints are complex.
Systematic approaches help.

Q. How many n-bit numbers are there?

A. 1st bit 0 or 1, then 2nd bit 0 or 1, then ...

\[ A, n_1 = 2 \]
\[ B, n_2 = 2 \]
\[ C, n_3 = 2 \]
\[ 2 \times 2 \times ... \times 2 = 2^n \]
Q. How many subsets of a set of size $n$ are there?

A. $1^{st}$ member in or out; $2^{nd}$ member in or out, ... ⇒ $2^n$

Tip: Visualize an order in which decisions are being made

Q. How many 4-character passwords are there, if each character must be one of a, b, ..., z, 0, 1, ..., 9?

A. $36 \cdot 36 \cdot 36 \cdot 36 = 1,679,616 \approx 1.7$ million

Q. Ditto, but no character may be repeated?

A. $36 \cdot 35 \cdot 34 \cdot 33 = 1,413,720 \approx 1.4$ million

Q. How many arrangements of $n$ distinct items are possible?

A. $n \cdot (n-1) \cdot (n-2) \cdot ... \cdot 1 = n!$ (n factorial)

Q. How many permutations of DAWGY are there?

A. $5! = 120$

Q. How many of DAGGY?

A. $5!/2! = 60$

Q. How many of GODOGGY?

A. $\frac{7!}{3!2!1!1!} = 420$
Q. Your elf-lord avatar can carry 3 objects chosen from
1. sword
2. knife
3. staff
4. water jug
5. iPad w/magic WiFi
How many ways can you equip him/her?

\[ \frac{5 \cdot 4 \cdot 3}{3!} = \frac{5!}{3! \cdot 2!} = 10 \]

ordered ways in which to pick objects

but picking abc is equiv to acb, and bca, and ...

Combinations: number of ways to choose \( r \) unordered things from \( n \) distinct things

“n choose r” aka binomial coefficients

\[ \binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots1} = \frac{n!}{r!(n-r)!} \]

Important special case: how many (unordered) pairs from \( n \) objects

\[ \binom{n}{2} = \frac{n(n-1)}{2} = \Theta(n^2) \]

Combinations: examples

Q. How many different poker hands are possible (i.e., 5 cards chosen from a deck of 52 distinct possibilities)?

A. \( \binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2,598,960 \)

Q. 10 people meet at a party. If everyone shakes hands with everyone else, how many handshakes happen?

A. \( \binom{10}{2} = \frac{10 \cdot 9}{2 \cdot 1} = 45 \)

Many Identities. E.g.:
Combinatorial argument:
Let $S$ be a set of objects.
Show how to count the set one way $\rightarrow N$
Show how to count the set another way $\rightarrow M$
Conclude that $N=M$

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\]

proof 1: induction ...
proof 2: counting –
\((x+y) \cdot (x+y) \cdot (x+y) \cdot \ldots (x+y)\)
pick either $x$ or $y$ from each factor
How many ways did you get exactly $k$ $x$’s? \(\binom{n}{k}\)

another identity w/ binomial coefficients
\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

Proof:
\[
\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n
\]

another binomial theorem question

coefficient of $y^3$ in $(7x + 3y)^5$ ?

 Relevant term: \(\binom{5}{2} (7x)^2 (3y)^3\)
Coefficient: \(\binom{5}{2} (7x)^2 3^3\)
another general counting rule: inclusion-exclusion

If two sets or events $A$ and $B$ are disjoint, aka mutually exclusive, then

$$|A \cup B| = |A| + |B|$$

More generally, for two sets or events $A$ and $B$, whether or not they are disjoint,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

inclusion-exclusion

inclusion-exclusion in general

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap C| - |A \cap B| + |A \cap B \cap C|$$

General: + singles - pairs + triples - quads + ...

denumeration

example

How many of $1, 2, \ldots, 10$ are divisible by $2$, $3$, and/or $5$?

Let

$E_2 = \{x \mid 1 \leq x \leq 10 \land x \text{ is a multiple of } 2\}$

$E_3 = \{x \mid 1 \leq x \leq 10 \land x \text{ is a multiple of } 3\}$

$E_5 = \{x \mid 1 \leq x \leq 10 \land x \text{ is a multiple of } 5\}$

$$|E_2 \cup E_3 \cup E_5| = |E_2| + |E_3| + |E_5| - |E_2 \cap E_3| - |E_2 \cap E_5| - |E_3 \cap E_5| + |E_2 E_3 E_5|$$

$$= \frac{10}{2} + \frac{10}{3} + \frac{10}{5} - \frac{10}{2 \cdot 3} - \frac{10}{2 \cdot 5} - \frac{10}{3 \cdot 5} + \frac{10}{2 \cdot 3 \cdot 5}$$

$$= 5 + 3 + 2 - 1 - 1 - 0 + 0$$

$$= 8$$

more counting: the pigeonhole principle
If there are \( n \) pigeons in \( k \) holes and \( n > k \), then some hole contains more than one pigeon.

More precisely, some hole contains at least \( \lceil n/k \rceil \) pigeons.

To solve a pigeonhole principle problem:
1. Define the pigeons
2. Define the pigeonholes
3. Define the mapping of pigeons to pigeonholes

There are two people in London who have the same number of hairs on their head.

Typical head \( \sim 150,000 \) hairs
Londoners have between 0 and 999,999 hairs on their head.
Since there are more than 1,000,000 people in London…

Another example:

25 fleas sit on a 5 x 5 checkerboard, one per square. At the stroke of noon, all jump across an edge (not a corner) of their square to an adjacent square. Two must end up in the same square. Why?
Fleas on checkerboard

13 red squares, 12 black squares

Pigeons: fleas on red squares
Pigeonholes: black squares
Pigeon -> pigeonhole: red square flea maps to black square it jumps to.

Product Rule: \( n \) outcomes for \( A \): \( \prod n_i \) in total (tree diagram)

Permutations:
- ordered lists of \( n \) objects, no repeats: \( n(n-1)\ldots 1 = n! \)
- ordered lists of \( r \) objects from \( n \), no repeats: \( n!/(n-r)! \)

Combinations:
- “\( n \) choose \( r \),” aka binomial coefficients,
  unordered lists of \( r \) objects from \( n \): \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \)

Binomial Theorem: \((x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\)

Inclusion-Exclusion: \(|A \cup B| = |A| + |B| - |A \cap B|\)

Pigeonhole Principle