If $X$ and $Y$ are independent Poisson random variables with respective parameters $\lambda_1$ and $\lambda_2$, calculate the conditional distribution of $X$, given that $X + Y = n$.

$$p_{X|Y}(x|y) = \frac{P(X = x | Y = y)}{P(Y = y)}$$

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$$P(X = k | X + Y = n) = \frac{P(X = k, X + Y = n)}{P(X + Y = n)}$$

$$= \frac{P(X = k, Y = n - k)}{P(X + Y = n)}$$

$$= \frac{P(X = k)P(Y = n - k)}{P(X + Y = n)}$$

Show that $X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$

Proof:

$$P(X + Y = n) = \sum_{k=0}^{n} P(X = k, Y = n - k)$$

$$= \sum_{k=0}^{n} e^{-\lambda_1 - \lambda_2} \frac{\lambda_1^k}{k!} \frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= e^{-\lambda_1 - \lambda_2} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k! (n-k)!}$$

The conditional distribution of $X$, given that $X + Y = n$ is:

Binomial $(n, \frac{\lambda_1}{\lambda_1 + \lambda_2})$
Law of Total Expectation

X random variable on a sample space S

partition of S

\[ E(X) = \sum \sum x \Pr(X = x | A_i) Pr(A_i) \]

\[ = \sum \sum x \Pr(X = x | A_i) Pr(A_i) \]

\[ = \sum x \sum_i \Pr(X = x | A_i) Pr(A_i) \]

\[ = \sum x \Pr(X = x) \]

Law of Total Expectation

X random variable on a sample space S

partition of S

Version with conditional distributions

\[ E(X) = \sum_y E(X | Y = y) Pr(Y = y) \]

E(X | Y = y) = \sum x \Pr(X = x | Y = y) = \sum x \Pr(X = x | Y = y)

Law of Total Expectation : Application

System that fails in step i independently with probability p

X # steps to fail

\[ E(X) \]

Let A be the event that system fails in first step.

\[ E(X) = E(X | A) Pr(A) + E(X | \overline{A}) Pr(\overline{A}) \]

\[ = p + (1 + E(X))(1 - p) \]

\[ = 1 + (1 - p)E(X) \]

\[ E(X) = \frac{1}{p} \]

Law of Total Expectation : Example

A miner is trapped in a mine containing 3 doors.

• D1: The 1st door leads to a tunnel that will take him to safety after 3 hours.
• D2: The 2nd door leads to a tunnel that returns him to the mine after 5 hours.
• D3: The 3rd door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters (12, 1/3).

At all times, he is equally likely to choose any one of the doors.

E(time to reach safety) ?

\[ E(T) = E(T | D_1) \frac{1}{3} + E(T | D_2) \frac{1}{3} + E(T | D_3) \frac{1}{3} \]

\[ = 3 \frac{1}{3} + 5 \frac{1}{3} + (4 + E(T)) \frac{1}{3} \]

\[ = 6 \]

\[ E(T) = 6 \]
Problem

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all the passengers.

\[
X \quad \text{number of people who enter} \\
Y \quad \text{number of stops} \\
E(Y|X=k) = \sum_{i=1}^{N} \frac{1}{N} \
E(Y|X=k) = E(Y_1 + \ldots + Y_N|X=k) \\
Y_i \text{ indicates a stop on floor } i \\
E(Y_i|X=k) = \left(1 - \left(1 - \frac{1}{N}\right)^k\right) \\
Pr(X=k) = e^{-10} \frac{10^k}{k!} \\
E(Y) = \sum_{k=0}^{\infty} E(Y|X=k)Pr(X=k) \\
E(Y|X=k) = E(Y_1 + \ldots + Y_N|X=k) \\
E(Y_i|X=k) = \left(1 - \left(1 - \frac{1}{N}\right)^k\right) \\
Pr(X=k) = e^{-10} \frac{10^k}{k!}
\]

Game of Craps

- Begin by rolling an ordinary pair of dice
- If the sum of dice is 2, 3 or 12, the player loses
- If the sum of dice is 7 or 11, the player wins
- If it is any other number, say k, the player continues to roll the dice until the sum is either 7 or k.
  - If it is 7, the player loses.
  - If it is k, the player wins.

Let R denote the number of rolls of the dice in a game of craps.
- What is \(E(R)\)?

Problem

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