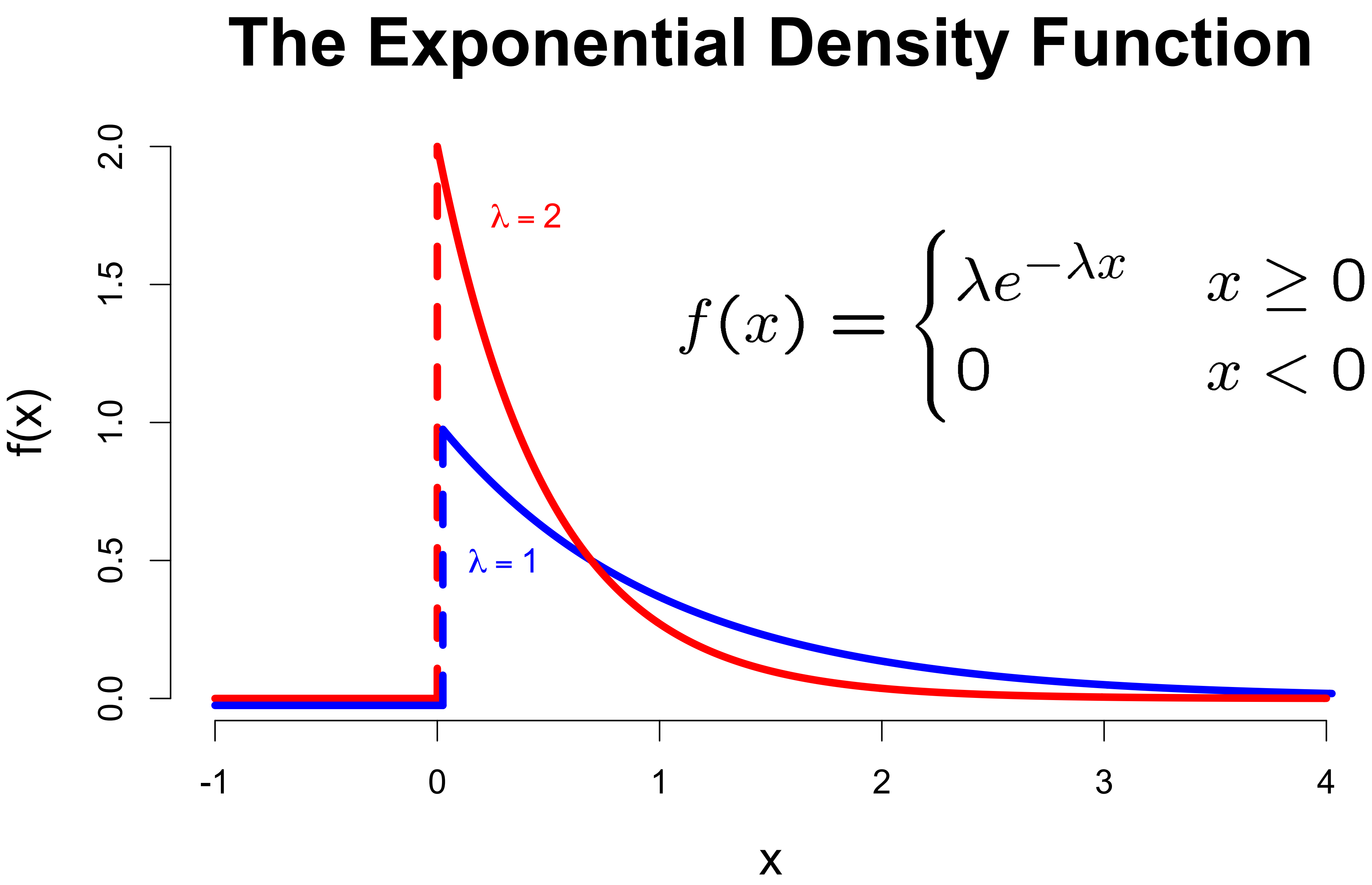
X ~ Exp(λ)


The Exponential Density Function
λ = 2
λ = 1
$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$
f(x)
x
0.0
0.5
1.0
1.5
2.0
-1
0
1
2
3
4

X ~ Exp(λ)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$E[X] = \frac{1}{\lambda} \qquad \mathrm{Var}[X] = \frac{1}{\lambda^2}$$

$$\Pr(X \geq t) = e^{-\lambda t} = 1\text{-}F(t)$$

Memorylessness:

$$\Pr(X > s + t \mid X > s) = \Pr(X > t)$$

Radioactive decay: How long until the next alpha particle?

Customers: how long until the next customer/packet arrives at the checkout stand/server?

Buses: How long until the next #71 bus arrives on the Ave?

Yes, they have a schedule, but given the vagaries of traffic, riders with-bikes-and-baby-carriages, etc., can they stick to it?

Relation to the Poisson:

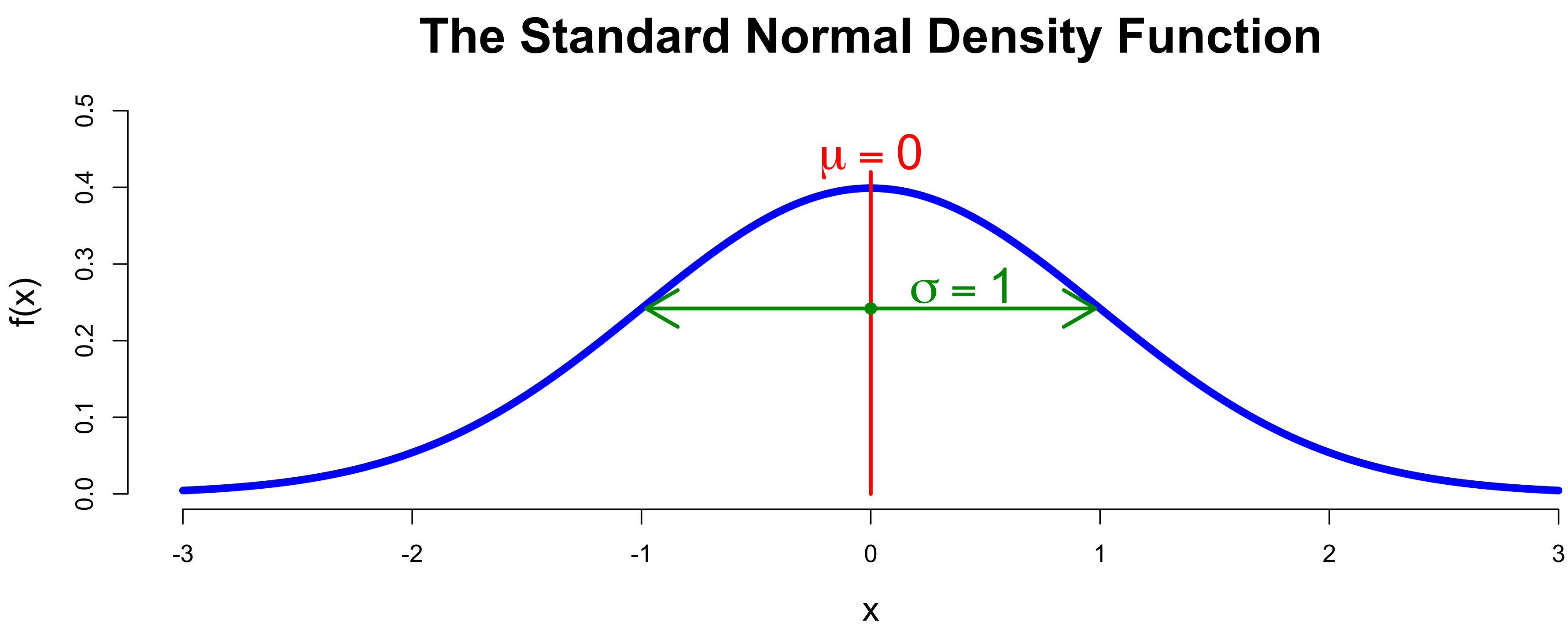
Poisson: *how many* events in a *fixed time*;
Exponential: *how long* until the *next event*

Relation to geometric: Geometric is discrete analog

X is a normal (aka Gaussian) random variable $X \sim N(\mu, \sigma^2)$

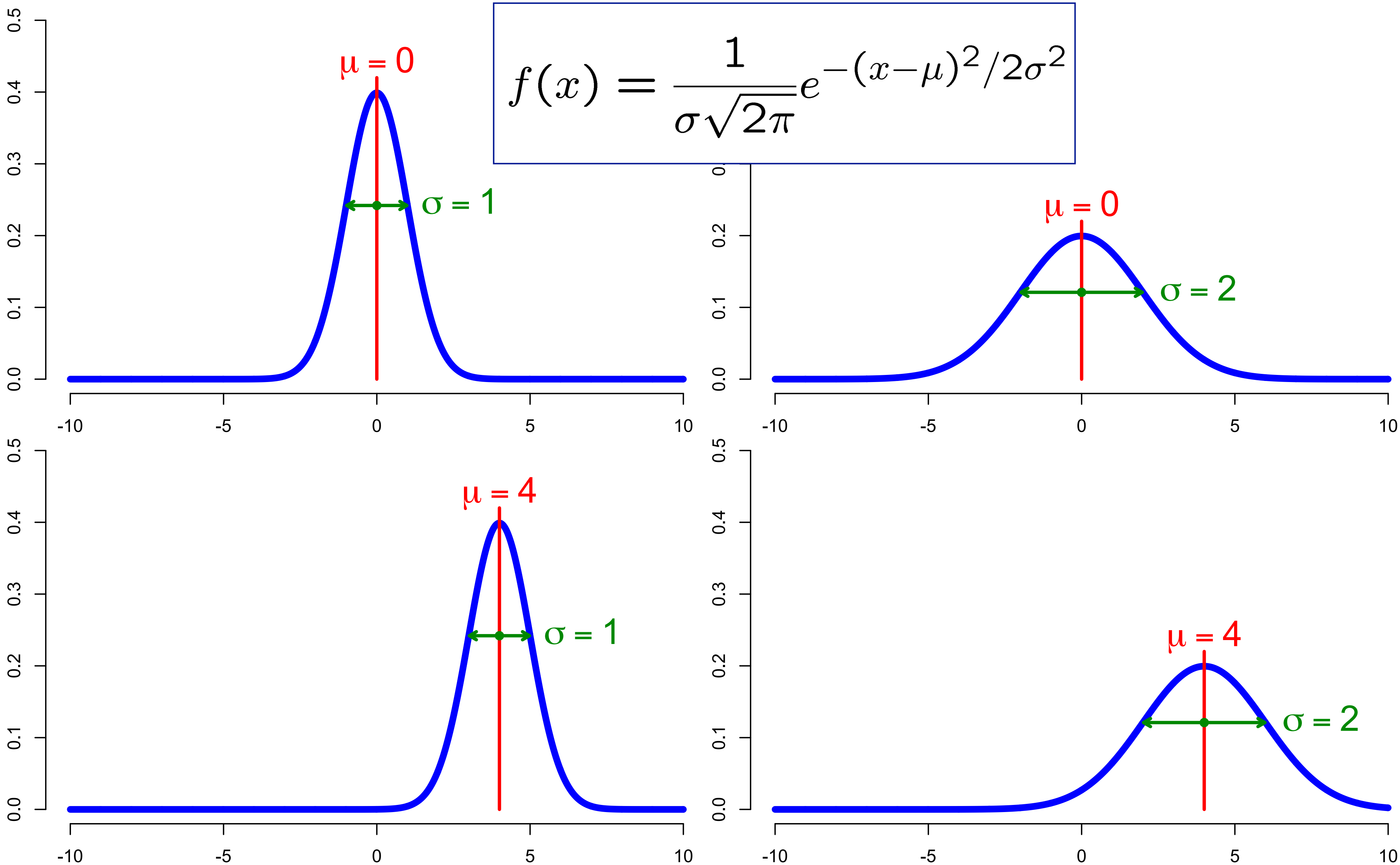
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$E[X] = \mu \qquad \mathrm{Var}[X] = \sigma^2$$



changing μ, σ

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$



density at μ is ≈ .399/σ

X is a normal random variable $X \sim N(\mu,\sigma^2)$

$Y = aX + b$

$E[Y] = E[aX+b] = a\mu + b$

$Var[Y] = Var[aX+b] = a^2\sigma^2$

$Y \sim N(a\mu + b, a^2\sigma^2)$

Important special case: $Z = (X-\mu)/\sigma \sim N(0,1)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$Z \sim N(0,1)$ "*standard* (or *unit*) normal"

Use $\Phi(z)$ to denote CDF, i.e.

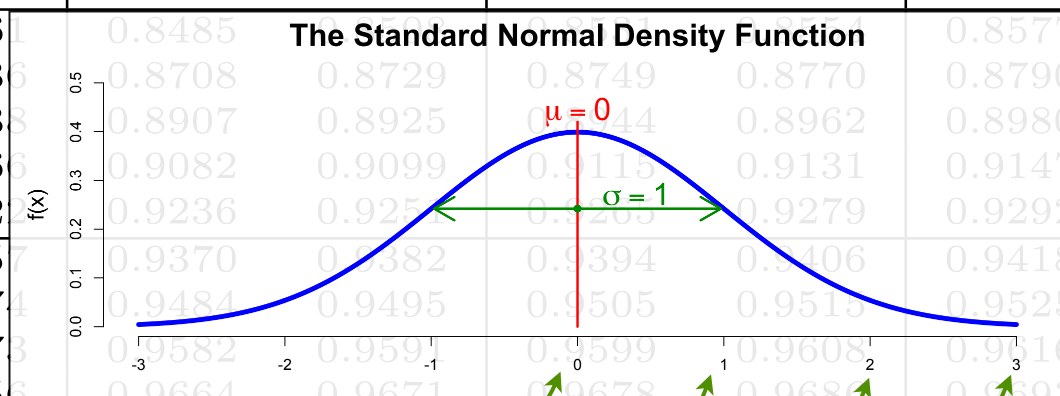
$$\Phi(z) = \Pr(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

no closed form ☹

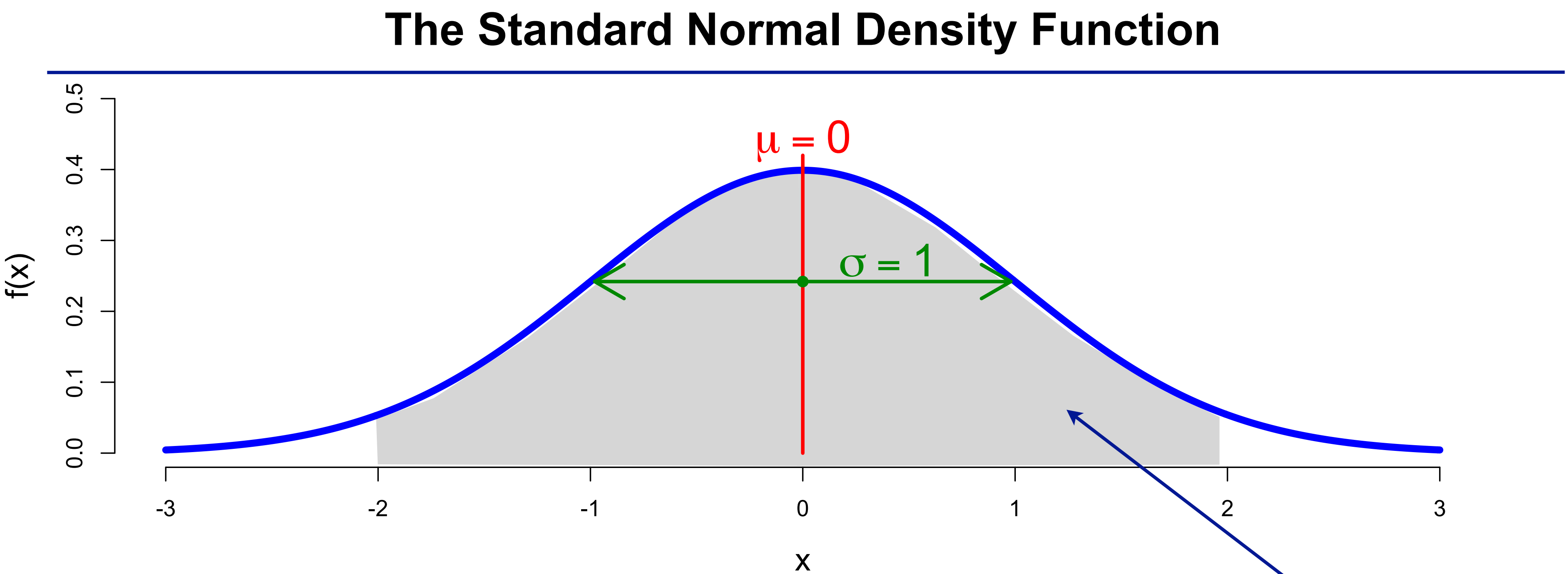
Table of the Standard Normal Cumulative Distribution Function $\Phi(Z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	[illegible]	0.7517
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	[illegible]	0.7823
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	[illegible]	[illegible]	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	[illegible]	[illegible]	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997

$\Phi(.46)$



E.g., see B&T p155, p531



If $Z \sim N(\mu,\sigma)$ what is $P(\mu-\sigma < Z < \mu+\sigma)$?

$P(\mu - \sigma < Z < \mu + \sigma) = \Phi(1) - \Phi(-1) \approx 68\%$

$P(\mu - 2\sigma < Z < \mu + 2\sigma) = \Phi(2) - \Phi(-2) \approx 95\%$

$P(\mu - 3\sigma < Z < \mu + 3\sigma) = \Phi(3) - \Phi(-3) \approx 99\%$

X ~ Bin(n,p)

E[X] = np   Var[X] = np(1-p)

Poisson approx: good for n large, p small (np constant)

Normal approx: For large n, (p stays fixed):

X ≈ Y ~ N(E[X],Var[X]) = N(np,np(1-p))

Normal approximation good when np(1-p) ≥ 10

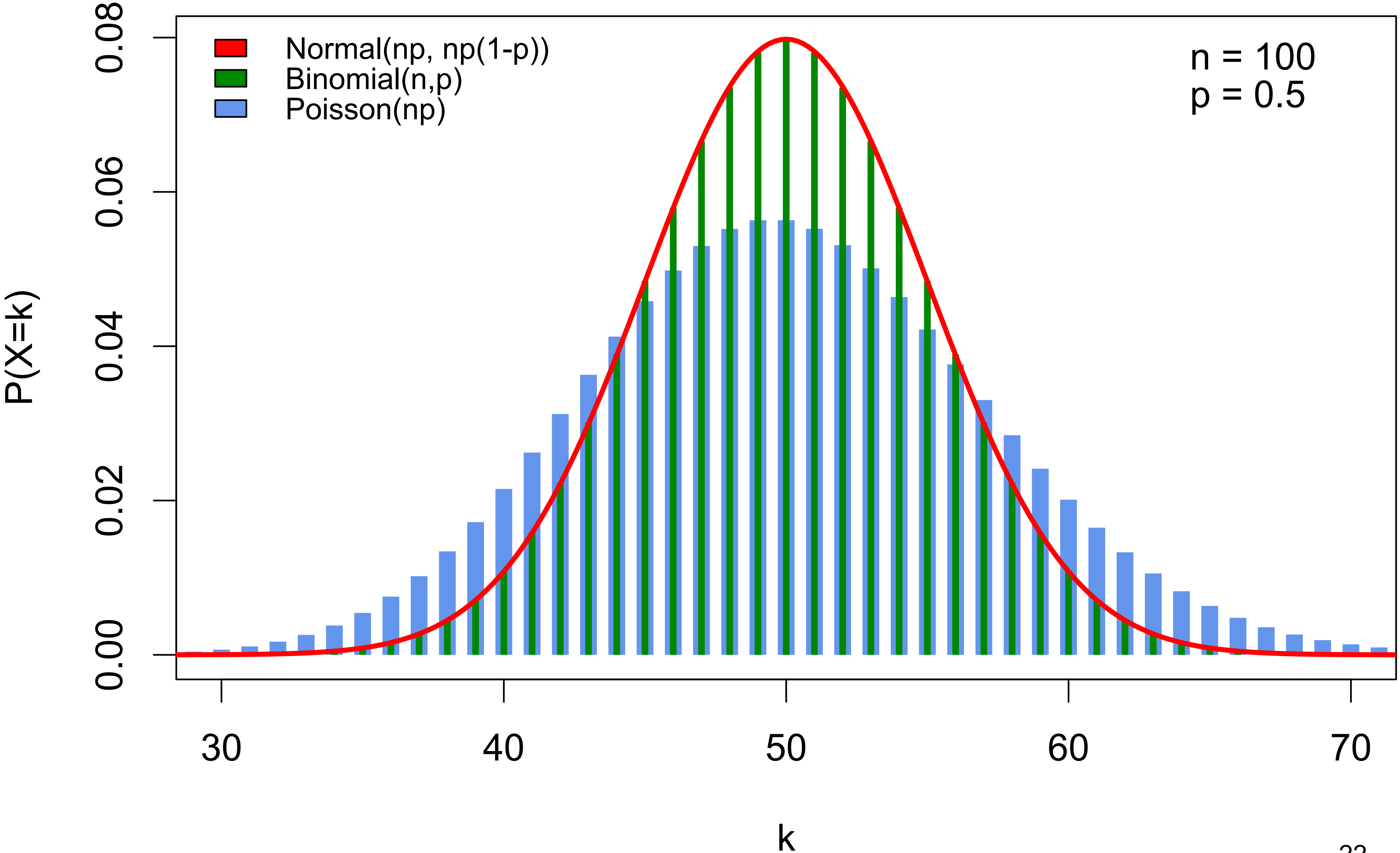
DeMoivre-Laplace Theorem:

Let S_n = number of successes in n trials (with prob. p).

Then, as $n \to \infty$:

$$Pr\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right) \longrightarrow \Phi(b) - \Phi(a)$$

normal approximation to binomial



Fair coin flipped 40 times. Probability of 20 heads?

Exact answer:

$$P(X = 20) = \binom{40}{20}\left(\frac{1}{2}\right)^{40} \approx \boxed{0.1254}$$

Normal approximation:

$$\begin{aligned} P(X = 20) &= P(19.5 \le X < 20.5) \\ &= P\left(\frac{19.5 - 20}{\sqrt{10}} \le \frac{X - 20}{\sqrt{10}} < \frac{20.5 - 20}{\sqrt{10}}\right) \\ &\approx P\left(-0.16 \le \frac{X - 20}{\sqrt{10}} < 0.16\right) \\ &\approx \Phi(0.16) - \Phi(-0.16) \approx \boxed{0.1272} \end{aligned}$$

Consider i.i.d. (independent, identically distributed) random vars $X_1, X_2, X_3, \ldots$

X_i has $\mu = E[X_i]$ and $\sigma^2 = Var[X_i]$

As $n \to \infty$,

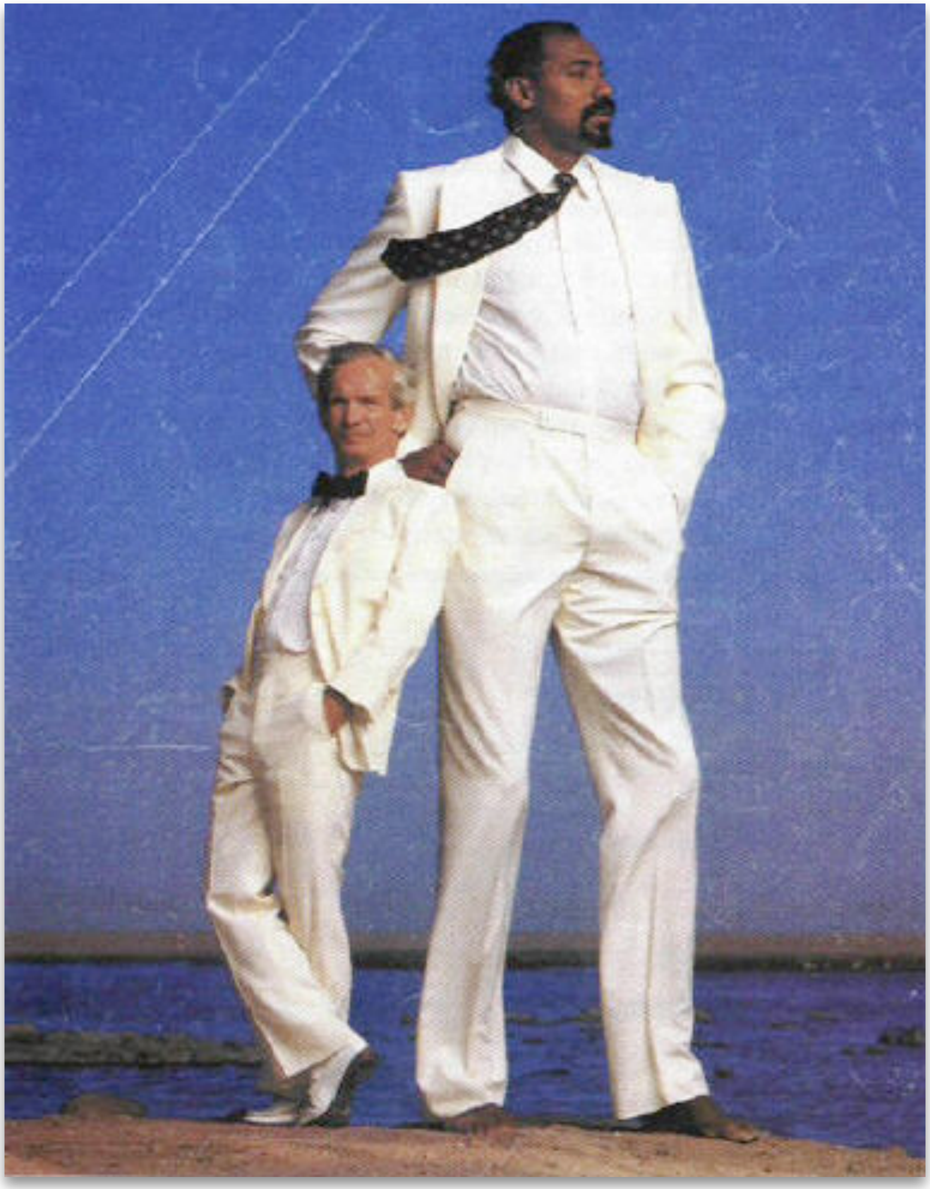
$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \longrightarrow N(0,1)$$

Restated: As $n \to \infty$,

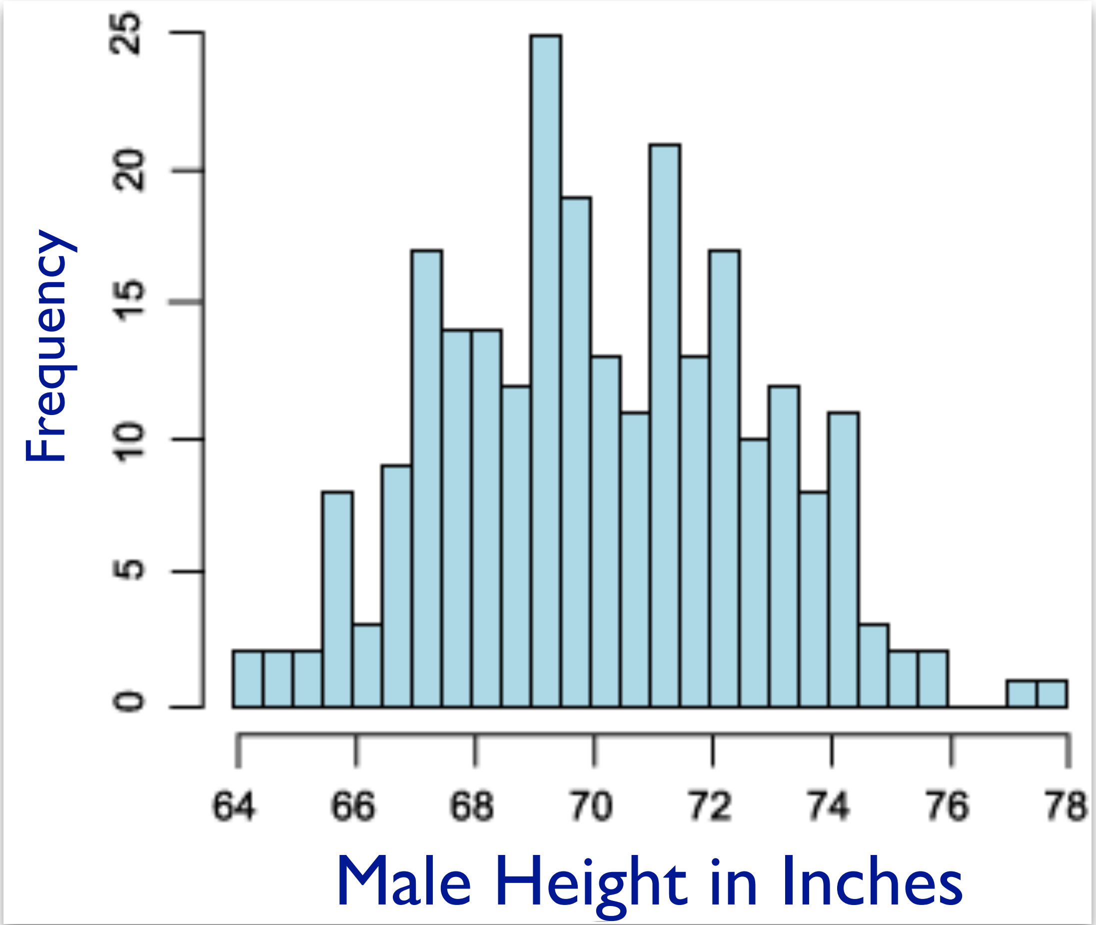
$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

How tall are you? Why?

Credit: Annie Leibovitz, © 1987 ?



Willie Shoemaker & Wilt Chamberlain

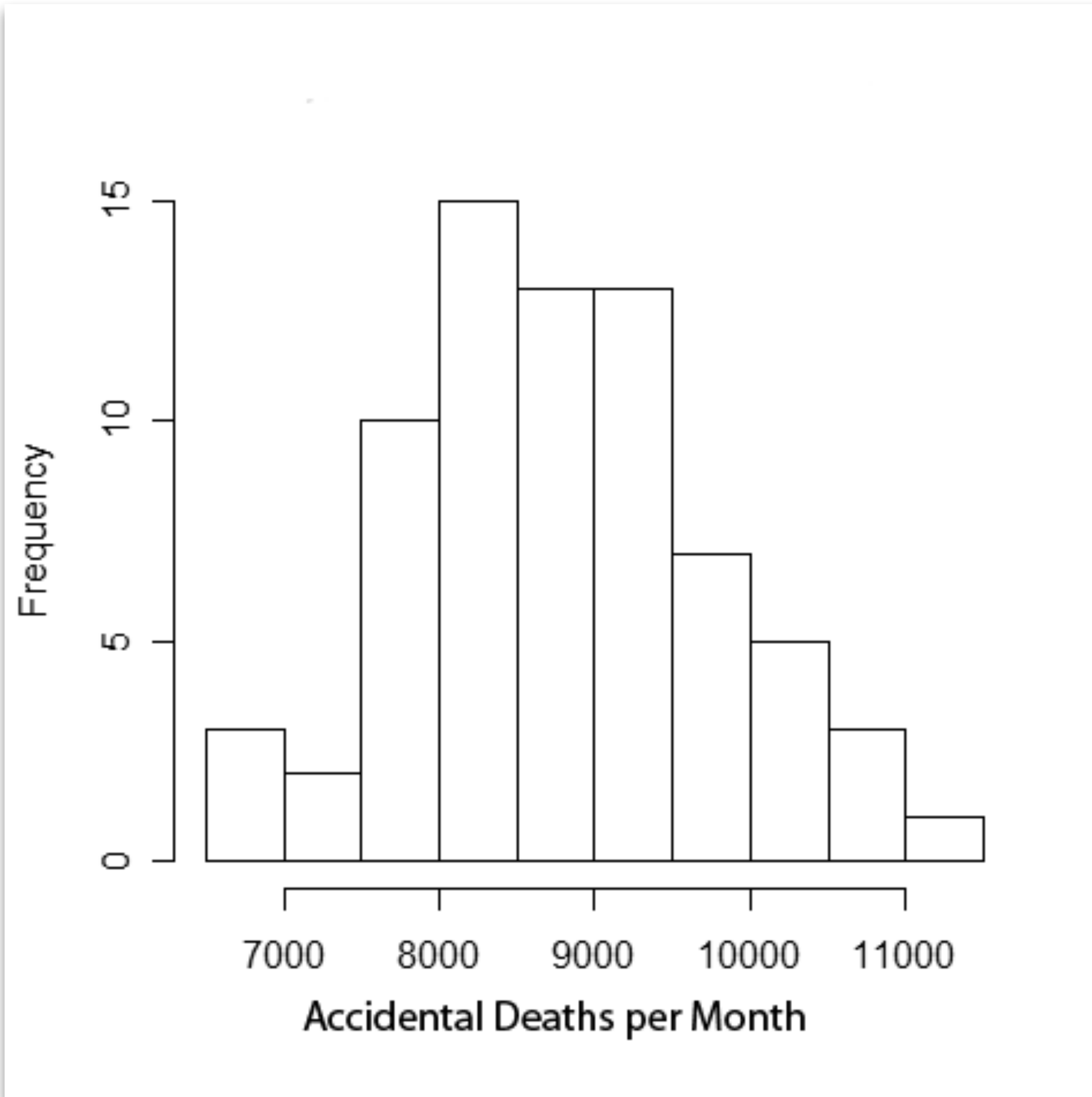


Human height is approximately normal.

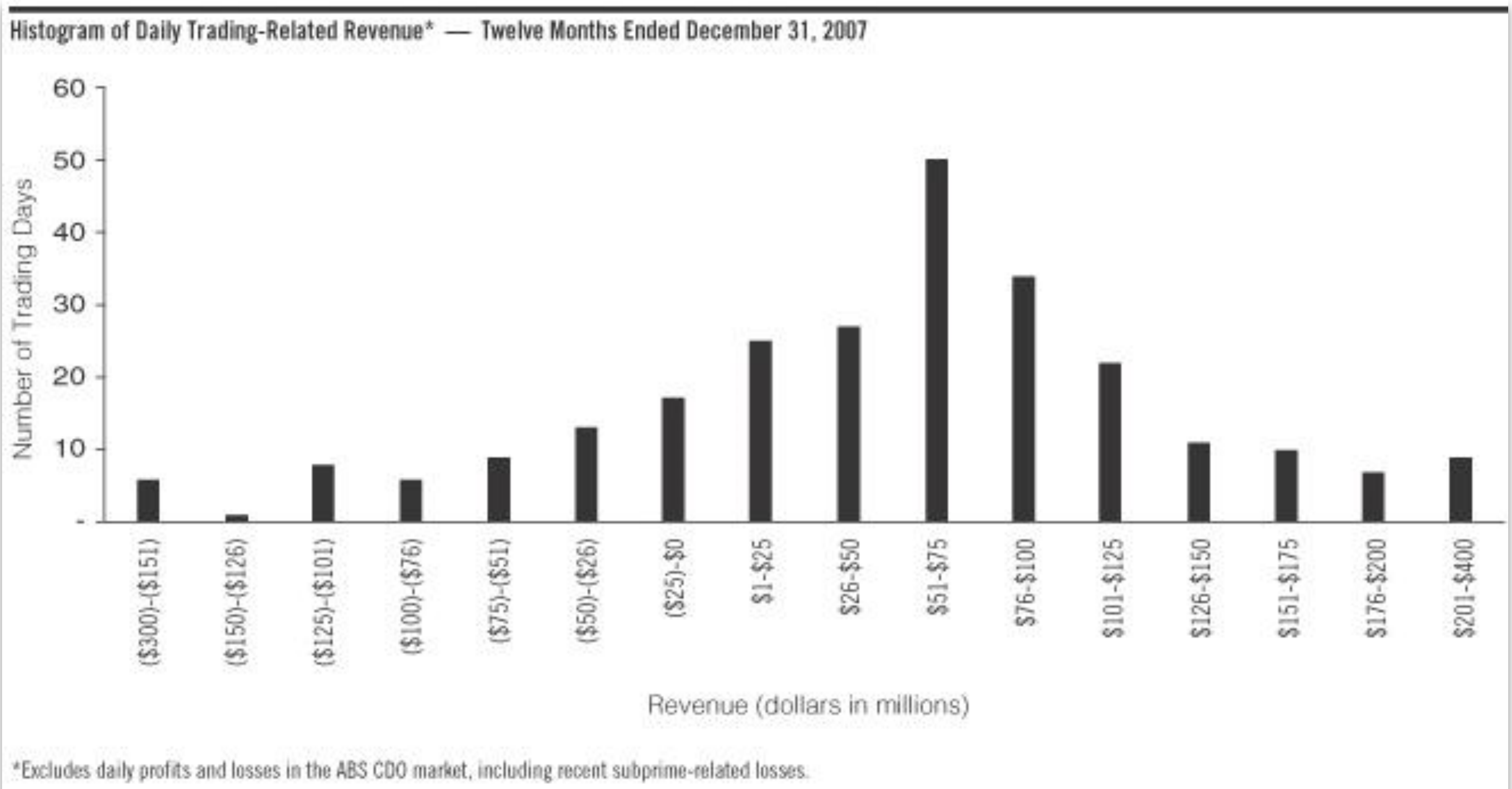
Why might that be true?

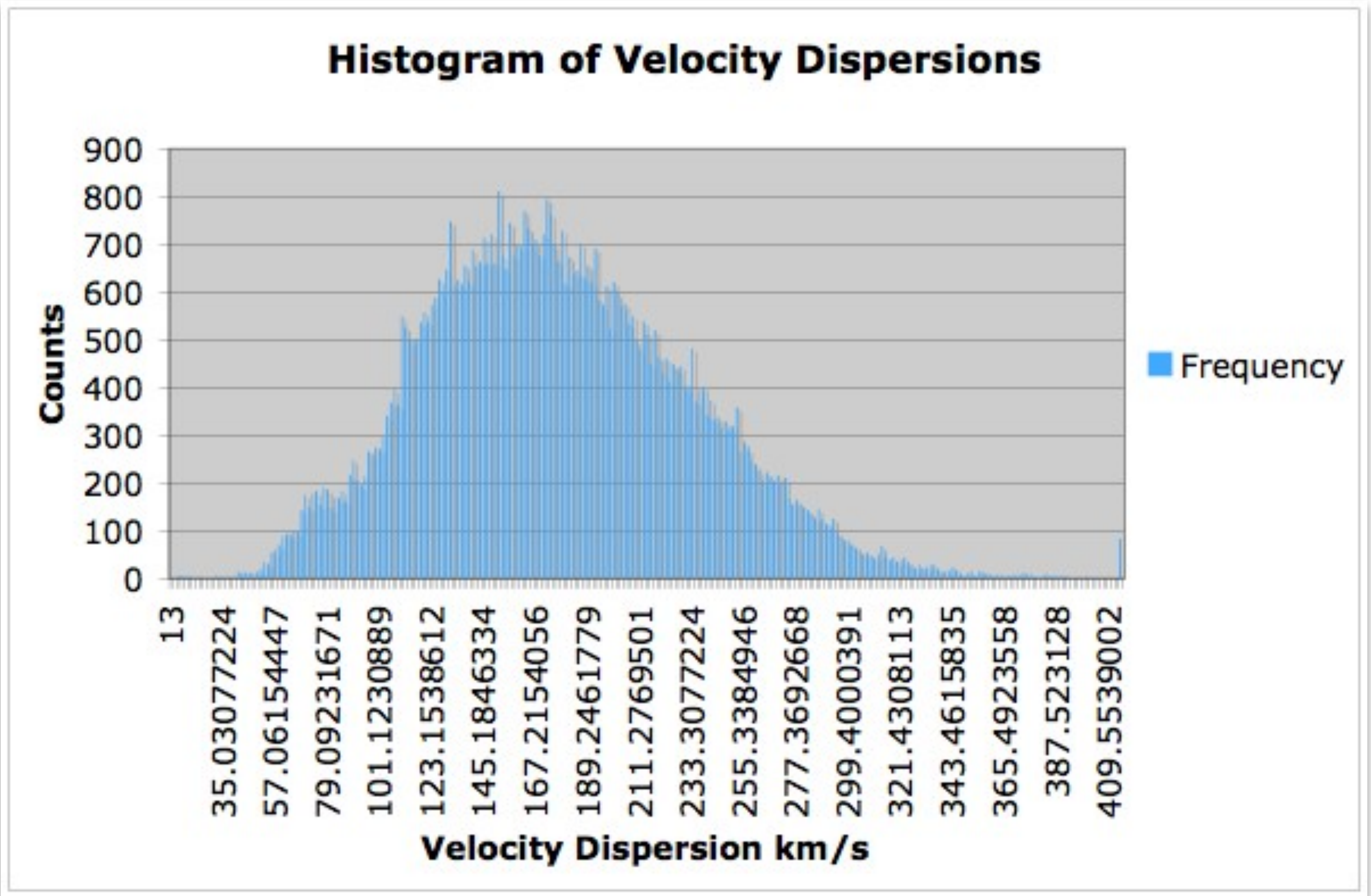
R.A. Fisher (1918) noted it would follow from CLT if height were the sum of many independent random effects, e.g. many genetic factors (plus some environmental ones like diet). I.e., suggested part of *mechanism* by looking at *shape* of the curve. (WAY before anyone really knew what genes were...)

in the real world…



in the real world…




Histogram of Velocity Dispersions
Counts
900
800
700
600
500
400
300
200
100
0
13
35.030772224
57.061544447
79.092231671
101.12230889
123.15386612
145.18463334
167.21540556
189.24617779
211.27695001
233.30772224
255.33849946
277.36926668
299.40003391
321.43081113
343.46158835
365.49233558
387.5231128
409.5539002
Velocity Dispersion km/s
Frequency

pdf and cdf

$f(x) = \frac{d}{dx} F(x)$ $\qquad$ $F(a) = \int_{-\infty}^{a} f(x)\, dx$

sums become integrals, e.g.

$E[X] = \Sigma_x\, xp(x)$ $\qquad$ $E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$

most familiar properties still hold, e.g.

$E[aX+bY+c] = aE[X]+bE[Y]+c$

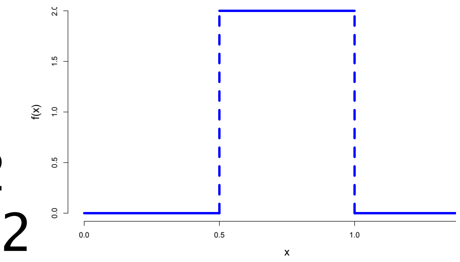
$Var[X] = E[X^2] - (E[X])^2$

Three important examples

X ~ Uni(α,β) uniform in [α,β]

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & x \in [\alpha, \beta] \\ 0 & \text{otherwise} \end{cases}$$

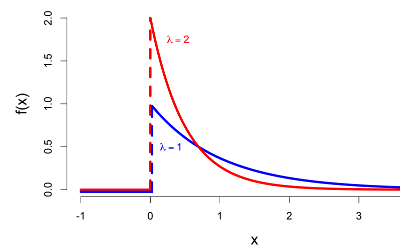
$$E[X] = (\alpha+\beta)/2$$
$$Var[X] = (\alpha-\beta)^2/12$$



X ~ Exp(λ) exponential

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$E[X] = \frac{1}{\lambda}$$
$$\mathrm{Var}[X] = \frac{1}{\lambda^2}$$



X ~ N(μ, σ²) normal (aka Gaussian)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$E[X] = \mu$$
$$\mathrm{Var}[X] = \sigma^2$$