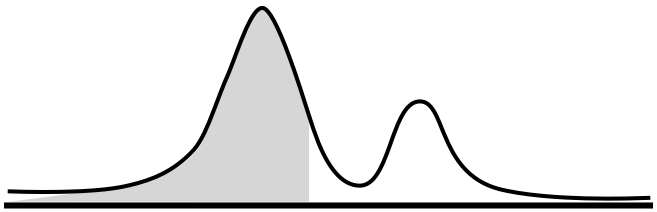
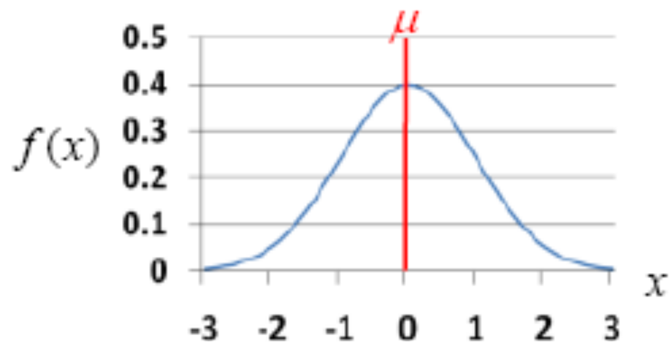
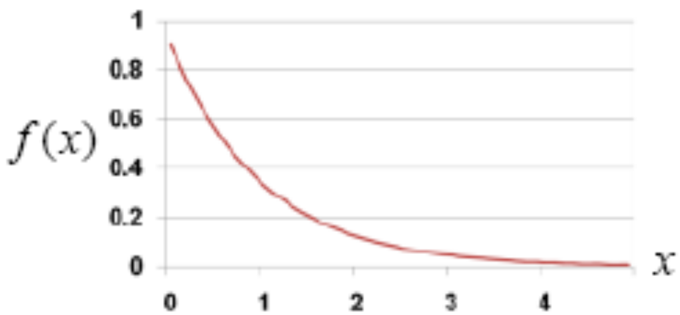
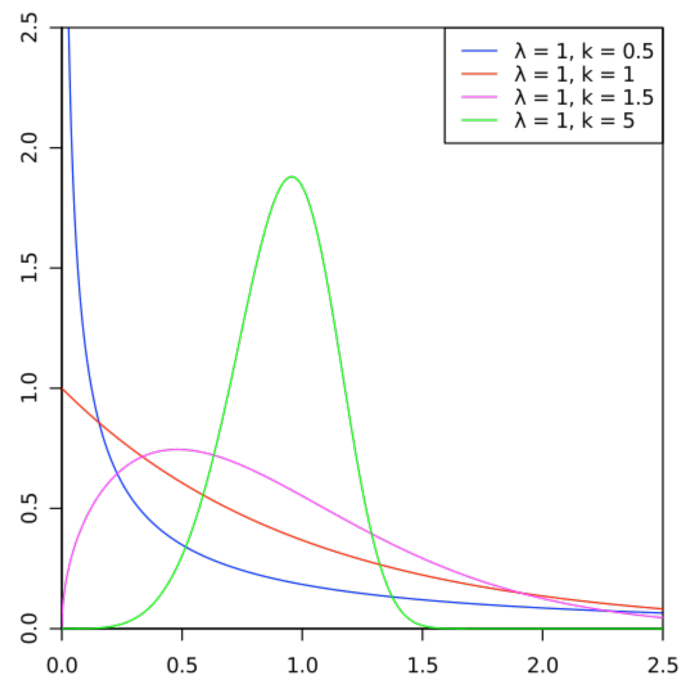




continuous random variables





Discrete random variable: takes values in a finite or countable set, e.g.

X $\in$ {1,2, ..., 6} with equal probability

X is positive integer i with probability 2^{-i}

Continuous random variable: takes values in an uncountable set, e.g.

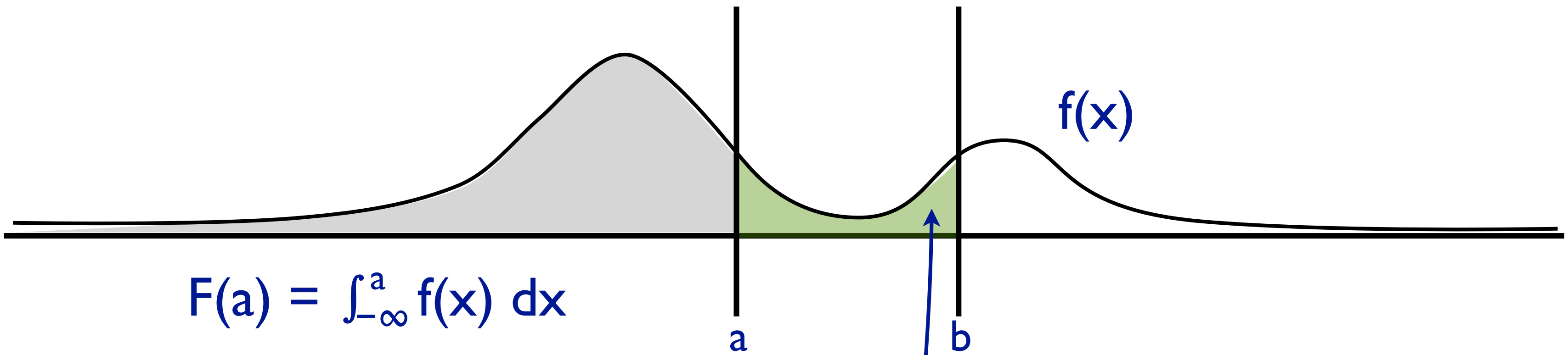
X is the weight of a random person (a real number)

X is a randomly selected point inside a unit square

X is the waiting time until the next packet arrives at the server

pdf and cdf

f(x) : the *probability density function* (or simply "density")



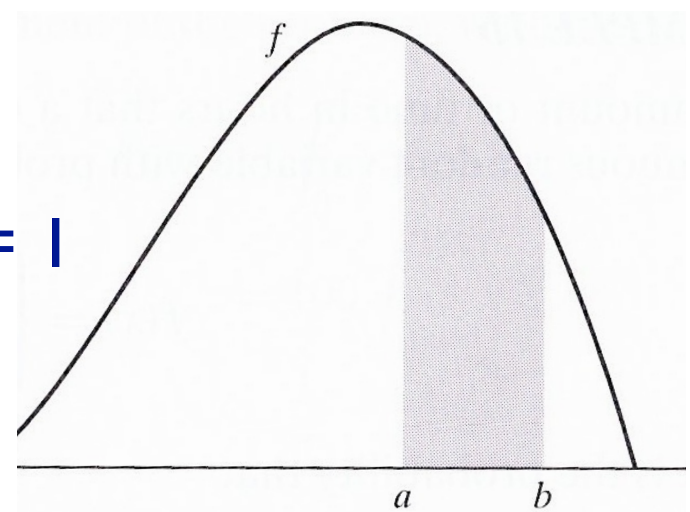
$P(X \leq a) = F(x)$: the *cumulative distribution function* (or simply "distribution")

$P(a < X \leq b) = F(b) - F(a)$

Need $f(x) \geq 0$, & $\int_{-\infty}^{+\infty} f(x)\, dx \;\; (= F(+\infty)) \;\; = 1$

A key relationship:

$f(x) = \frac{d}{dx} F(x)$, since $F(a) = \int_{-\infty}^{a} f(x)\, dx$,



Densities are *not* probabilities

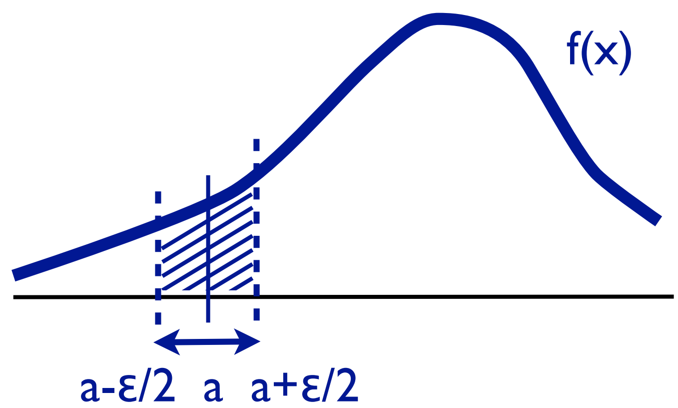
$P(X = a) = P(a \le X \le a) = F(a)\text{-}F(a) = 0$

I.e., the probability that a continuous random variable falls *at* a specified point is *zero*

$P(a - \varepsilon/2 \le X \le a + \varepsilon/2) =$

$F(a + \varepsilon/2) - F(a - \varepsilon/2)$

$\approx \varepsilon \cdot f(a)$



I.e., The probability that it falls *near* that point is proportional to the density; in a large random sample, expect more samples where density is higher (hence the name "density").

Much of what we did with discrete r.v.s carries over almost unchanged, with $\Sigma_x \ldots$ replaced by $\int \ldots dx$

E.g.

For discrete r.v. X, $E[X] = \Sigma_x \, xp(x)$

For continuous r.v. X, $$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

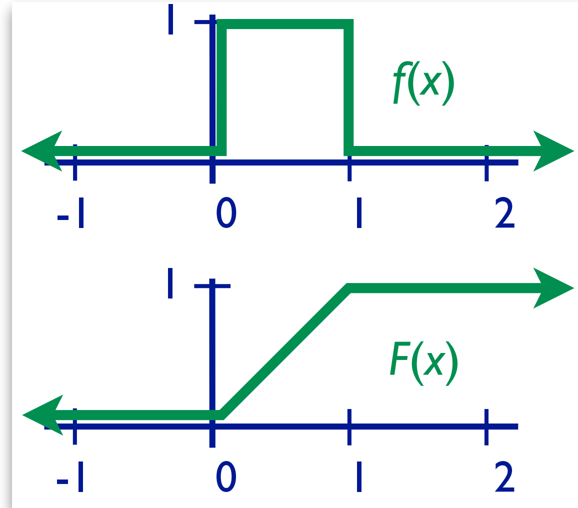
Why?

(a) We define it that way

(b) The probability that X falls "near" x, say within $x \pm dx/2$, is $\approx f(x)dx$, so the "average" X should be $\approx \Sigma\, xf(x)dx$ (summed over grid points spaced dx apart on the real line) and the limit of that as $dx \to 0$ is $\int xf(x)dx$

example

Let $f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$



$$\begin{aligned} F(a) &= \int_{-\infty}^{a} f(x)dx \\ &= \begin{cases} 0 & \text{if } a \leq 0 \\ a & \text{if } 0 < a \leq 1 \text{ (since } a = \int_0^a 1dx) \\ 1 & \text{if } 1 < a \end{cases} \end{aligned}$$

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$\text{Var}[X] = E[X^2] - (E[X])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (\sigma \approx 0.29)$

Linearity

$$E[aX+b] = aE[X]+b$$

$$E[X+Y] = E[X]+E[Y]$$

still true, just as for discrete

Functions of a random variable

$$E[g(X)] = \int g(x)f(x)dx$$

just as for discrete, but w/integral

Definition is same as in the discrete case

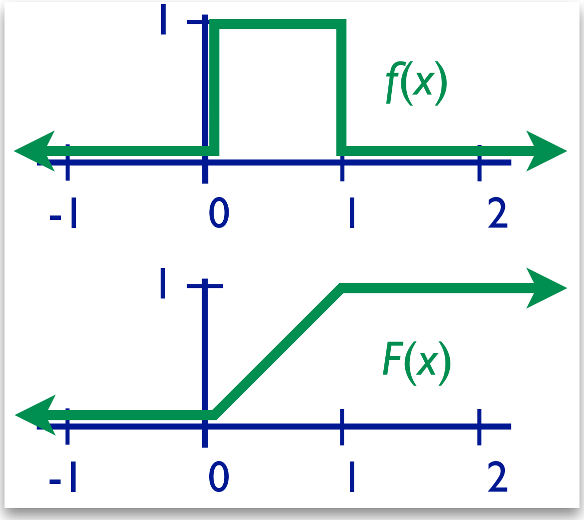
$Var[X] = E[(X-\mu)^2]$ where $\mu = E[X]$

Identity still holds:

$Var[X] = E[X^2] - (E[X])^2$ proof "same"

example

Let $f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$



$$F(a) = \int_{-\infty}^{a} f(x)dx$$

$$= \begin{cases} 0 & \text{if } a \le 0 \\ a & \text{if } 0 < a \le 1 \text{ (since } a = \int_0^a 1dx) \\ 1 & \text{if } 1 < a \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)dx = \int_0^1 x\, dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{1}{12} \quad (\sigma \approx 0.29)$$

continuous random variables: summary

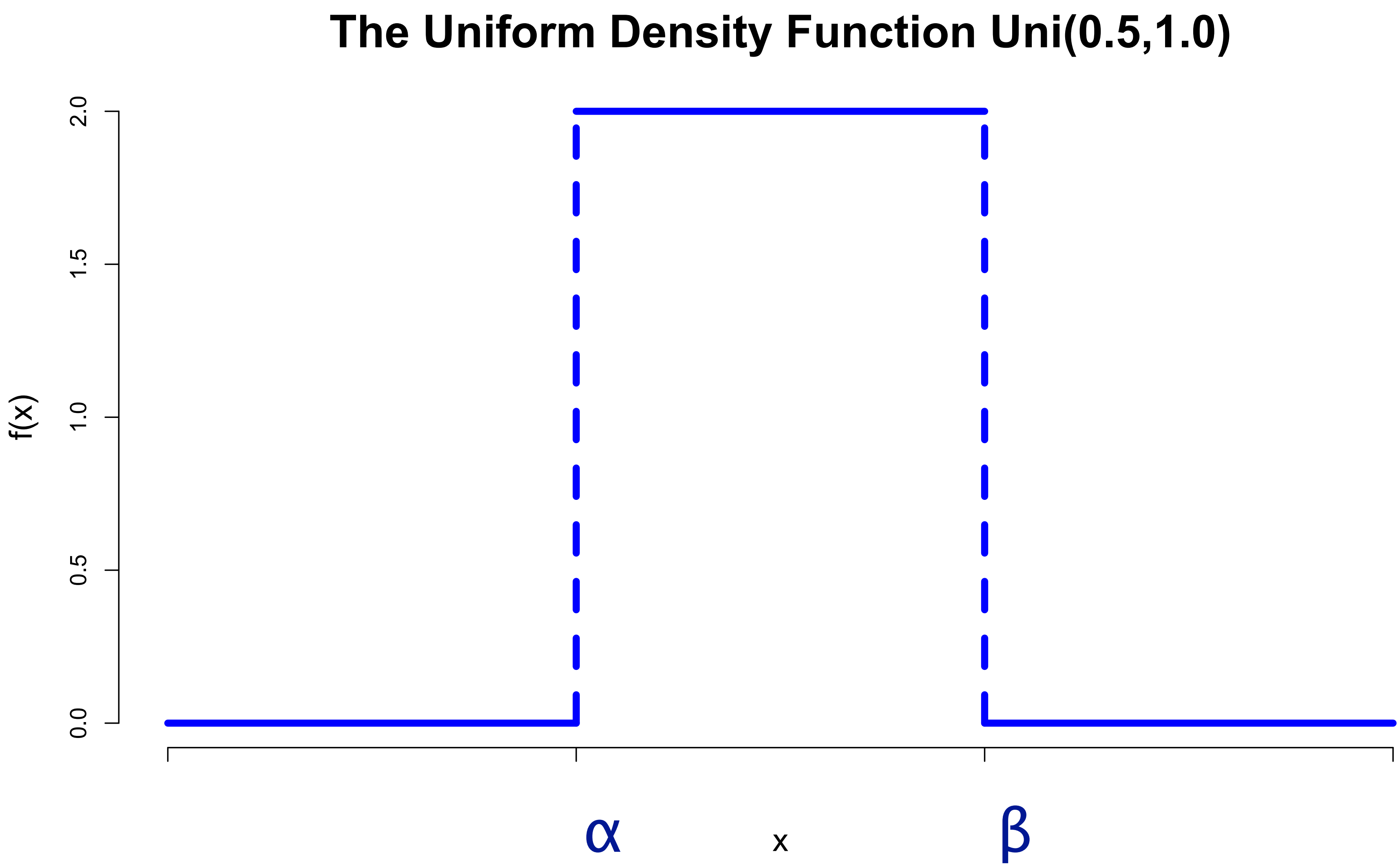
Continuous random variable X has density f(x), and

$$\Pr(a \leq X \leq b) = \int_a^b f(x)\,dx$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

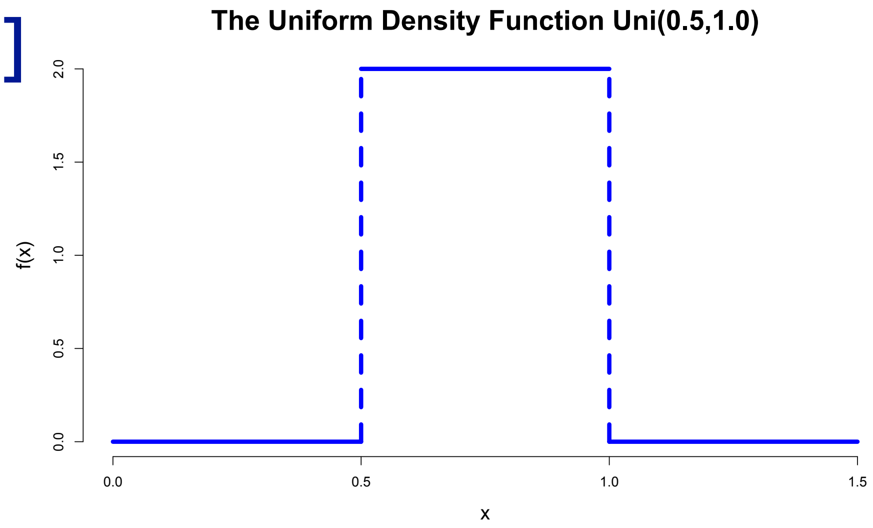
$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx$$

X ~ Uni(α,β) is uniform in [α,β] $f(x) = \begin{cases} \frac{1}{\beta-\alpha} & x \in [\alpha, \beta] \\ 0 & \text{otherwise} \end{cases}$



uniform random variable

X ~ Uni(α,β) is uniform in [α,β]



$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & x \in [\alpha,\beta] \\ 0 & \text{otherwise} \end{cases}$$

$$\Pr(a \le X \le b) = \int_a^b f(x)\,dx = \frac{b-a}{\beta-\alpha}$$

if α≤a≤b≤β:

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \frac{\alpha+\beta}{2}$$