Randomized Algorithms

Analyzing Algorithms

Goal: "Runs fast on typical real problem instances"

How do we evaluate this?

Example: Binary search

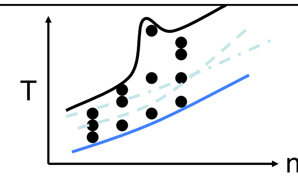
Given a sorted array, determine if the array contains the number 157?

Measuring efficiency

Time ≈ # of instructions executed in a simple programming language

- only simple operations (+,*,-,=,if,call,…)
- each operation takes one time step
- each memory access takes one time step

Complexity analysis



Problem size n

- Best-case complexity: min # steps algorithm takes on any input of size n
- Average-case complexity: avg # steps algorithm takes on inputs of size n
- **Worst-case complexity**: max # steps algorithm takes on any input of size n

Complexity

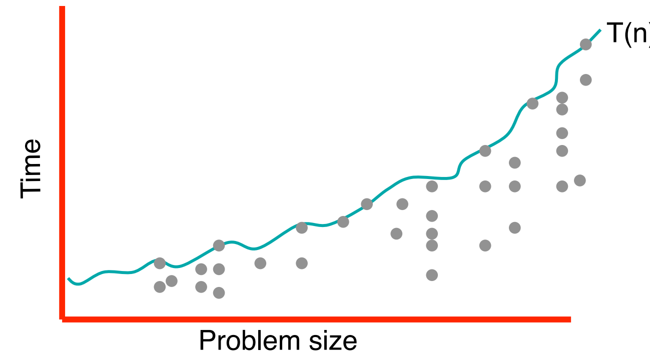
The *complexity* of an algorithm associates a number *T(n)*, the worst-case time the algorithm takes on problems of size *n*, with each problem size *n*.

Mathematically,

T: N+ → R+

I.e., *T* is a function that maps positive integers (problem sizes) to positive real numbers (number of steps).

Complexity



Simple Example

Array of bits.

I promise you that either they are all 1's or ½ 0's and ½ 1's.

Give me a program that will tell me which it is.

Best case?

Worst case?

Neat idea: use randomization to reduce the worst case

Complexity

The *complexity* of an algorithm associates a number *T(n)*, the worst-case time the algorithm takes on problems of size *n*, with each problem size *n*.

For **randomized algorithms**, look at worst-case value of E(T), where the expectation is taken over randomness in algorithm.

Quicksort

Sorting algorithm (assume for now all elements distinct)

Given array of some length n
If n = 0 or 1, halt

Else pick element p of array as "pivot"
  Split array into subarrays <p, > p
  Recursively sort elements < p
  Recursively sort elements > p

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Analysis of Quicksort

Worst case number of comparisons: $\binom{n}{2}$

How can we use randomization to improve running time?

  Pick random element as a pivot each step

  => **Randomized algorithm**

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Analysis of Randomized Quicksort

Quicksort with random pivots

X = # of comparisons.

$$X = \sum_{1 \le i < j \le n} X_{ij}$$

What is condition for elements i^{th} smallest and j^{th} smallest to get directly compared?

Claim: fate determined first time an elt in $[e_i, e_j]$ picked.

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Analysis of Randomized Quicksort
Fix pair i,j. Compute $E(X_{ij})$

Define A_k indicator r.v. that is 1 if elt in $[e_i, e_j]$ **first** selected at k^{th} level of tree

$$E(X_{ij}) = Pr(X_{ij} = 1)$$
$$= \sum_{1 \le k \le n} Pr(X_{ij} = 1 | A_k) Pr(A_k)$$
$$= \frac{2}{j - i + 1} \sum_{1 \le k \le n} Pr(A_k) = \frac{2}{j - i + 1}$$

$$Pr(X_{ij} = 1 | A_k) = \frac{2}{j - i + 1}$$

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Analysis of Randomized Quicksort

$$
\begin{aligned}
E(X) &= \sum_{1\le i<j\le n} E(X_{ij}) \\
&= \sum_{1\le i<n} \sum_{j>i} \frac{2}{j-i+1} \\
&\le 2 \sum_{1\le i<n} \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-i+1} \right) \\
&\le 2n\ln(n) + O(n)
\end{aligned}
$$

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