4: Discrete probability

Readings: BT 1.1-1.2, Rosen 6.1-6.2
Sample space: $S$ is the set of all possible outcomes of an experiment ($\Omega$ in your text book—Greek uppercase omega)

Coin flip: $S = \{\text{Heads, Tails}\}$

Flipping two coins: $S = \{(H,H), (H,T), (T,H), (T,T)\}$

Roll of one 6-sided die: $S = \{1, 2, 3, 4, 5, 6\}$

# emails in a day: $S = \{x : x \in \mathbb{Z}, x \geq 0\}$

YouTube hrs. in a day: $S = \{x : x \in \mathbb{R}, 0 \leq x \leq 24\}$
**Events:** \( E \subseteq S \) is some subset of the sample space

- Coin flip is heads: \( E = \{\text{Head}\} \)
- At least one head in 2 flips: \( E = \{(H,H), (H,T), (T,H)\} \)
- Roll of die is 3 or less: \( E = \{1, 2, 3\} \)
- # emails in a day < 20: \( E = \{ x : x \in \mathbb{Z}, \ 0 \leq x < 20 \} \)
- Wasted day (>5 YT hrs): \( E = \{ x : x \in \mathbb{R}, \ x > 5 \} \)
set operations on events

E and F are events in the sample space S
set operations on events

E and F are events in the sample space S

Event “E OR F”, written $E \cup F$

$S = \{1,2,3,4,5,6\}$
outcome of one die roll

$E = \{1,2\}, \ F = \{2,3\}$

$E \cup F = \{1, 2, 3\}$
E and F are events in the sample space $S$.

Event “E AND F”, written $E \cap F$ or $EF$.

$S = \{1, 2, 3, 4, 5, 6\}$

outcome of one die roll

$E = \{1, 2\}$, $F = \{2, 3\}$

$E \cap F = \{2\}$
set operations on events

E and F are events in the sample space S

EF = ∅ ⇔ E,F are “mutually exclusive”

S = \{1,2,3,4,5,6\}
outcome of one die roll

E = \{1,2\}, F = \{2,3\}, G = \{5,6\}
EF = \{2\}, not mutually exclusive, but E,G and F,G are
E and F are events in the sample space S

Event “not E,” written $\overline{E}$ or $\neg E$

$S = \{1, 2, 3, 4, 5, 6\}$
outcome of one die roll

$E = \{1, 2\}$
$\neg E = \{3, 4, 5, 6\}$
set operations on events

DeMorgan’s Laws

\[ \overline{E \cup F} = \overline{E} \cap \overline{F} \]

\[ \overline{E \cap F} = \overline{E} \cup \overline{F} \]
axioms of probability

Intuition: Probability as the relative frequency of an event

\[ \Pr(E) = \lim_{n \to \infty} \left( \frac{\text{# of occurrences of } E \text{ in } n \text{ trials}}{n} \right) \]

Axiom 1: \( 0 \leq \Pr(E) \leq 1 \)

Axiom 2: \( \Pr(S) = 1 \)

Axiom 3: If \( E \) and \( F \) are mutually exclusive (\( EF = \emptyset \)), then

\[ \Pr(E \cup F) = \Pr(E) + \Pr(F) \]

For any sequence \( E_1, E_2, \ldots, E_n \) of mutually exclusive events,

\[ \Pr \left( \bigcup_{i=1}^{n} E_i \right) = \Pr(E_1) + \cdots + \Pr(E_n) \]
implications of axioms

- \( \Pr(\bar{E}) = 1 - \Pr(E) \)

\[
\Pr(\bar{E}) = \Pr(S) - \Pr(E) \quad \text{because} \quad S = E \cup \bar{E}
\]

- If \( E \subseteq F \), then \( \Pr(E) \leq \Pr(F) \)

\[
\Pr(F') = \Pr(E) + \Pr(F - E) \geq \Pr(E)
\]

- \( \Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(EF) \)

inclusion-exclusion formula

- And many others
equally likely outcomes

Simplest case: sample spaces with equally likely outcomes.

Coin flips: \( S = \{\text{Heads, Tails}\} \)

Flipping two coins: \( S = \{(H,H),(H,T),(T,H),(T,T)\} \)

Roll of 6-sided die: \( S = \{1, 2, 3, 4, 5, 6\} \)

\[
\Pr(\text{each outcome}) = \frac{1}{|S|}
\]

In that case,

\[
\Pr(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|}
\]
Roll two 6-sided dice. What is \( \Pr(\text{sum of dice} = 7) \) ?

\[
S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\
    (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\
    (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\
    (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\
    (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\
    (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \} 
\]

\[
E = \{ (6,1), (5,2), (4,3), (3,4), (2,5), (1,6) \} 
\]

\[
\Pr(\text{sum} = 7) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}. 
\]
birthdays
What is the probability that, of $n$ people, none share the same birthday?

$|S| = (365)^n$

$|E| = (365)(364)(363)\cdots(365-n+1)$

$\Pr(\text{no matching birthdays}) = \frac{|E|}{|S|} = \frac{(365)(364)\cdots(365-n+1)}{(365)^n}$

Some values of $n$...

$n = 23$: $\Pr(\text{no matching birthdays}) < 0.5$

$n = 77$: $\Pr(\text{no matching birthdays}) < 1/5000$

$n = 100$: $\Pr(\text{no matching birthdays}) < 1/3,000,000$

$n = 150$: $\Pr(\text{...}) < 1/3,000,000,000,000,000,000$
n = 366?

Pr = 0

Above formula gives this, since

\[
\frac{(365)(364)\ldots(365-n+1)}{(365)^n} = 0
\]

when n = 366 (or greater).

Even easier to see via pigeon hole principle.
What is the probability that, of \( n \) people, none share the same birthday as you? 

\[
|S| = (365)^{n} \\
|E| = (364)^{n} \\
Pr(\text{no birthdays matches yours}) = \frac{|E|}{|S|} = \frac{(364)^n}{(365)^n}
\]

Some values of \( n \)…

\[
\begin{align*}
n &= 23: & Pr(\text{no matching birthdays}) & \approx 0.9388 \\
n &= 77: & Pr(\text{no matching birthdays}) & \approx 0.8096 \\
n &= 253: & Pr(\text{no matching birthdays}) & \approx 0.4995
\end{align*}
\]
poker hands
Consider 5 card poker hands.

A “straight” is 5 consecutive rank cards of any suit

What is $\Pr(\text{straight})$?

$|S| = \binom{52}{5}$

$|E| = 10 \cdot \binom{4}{1}^5$

$\Pr(\text{straight}) = \frac{10\binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394$
card flipping
52 card deck. Cards flipped one at a time.

After first ace (of any suit) appears, consider next card

Pr(next card = ace of spades) < Pr(next card = 2 of clubs) ?

Case 1: Take Ace of Spades out of deck

Shuffle remaining 51 cards, add ace of spades after first ace

|S| = 52! (all cards shuffled)

|E| = 51! (only 1 place ace of spades can be added)

Case 2: Do the same thing with the 2 of clubs

|S| and |E| have same size

So,

Pr(next = Ace of spades) = Pr(next = 2 of clubs) = 1/52
Theory is the same for a 3-card deck; \( \Pr = \frac{2}{3} = \frac{1}{3} \)
hats
n persons at a party throw hats in middle, select at random. What is \( \Pr(\text{no one gets own hat})? \)

\[
\Pr(\text{no one gets own hat}) = 1 - \Pr(\text{someone gets own hat})
\]

\[
\Pr(\text{someone gets own hat}) = \Pr(\bigcup_{i=1}^{n} E_i), \quad \text{where}
\]

\( E_i = \text{event that person } i \text{ gets own hat} \)

\[
\Pr(\bigcup_{i=1}^{n} E_i) = \sum_i P(E_i) - \sum_{i<j} \Pr(E_i \cap E_j) + \sum_{i<j<k} \Pr(E_i \cap E_j \cap E_k) \ldots
\]
Visualizing the sample space $S$:

<table>
<thead>
<tr>
<th>People:</th>
<th>P₁</th>
<th>P₂</th>
<th>P₃</th>
<th>P₄</th>
<th>P₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hats:</td>
<td>H₄</td>
<td>H₂</td>
<td>H₅</td>
<td>H₁</td>
<td>H₃</td>
</tr>
</tbody>
</table>

I.e., a sample point is a permutation $\pi$ of 1, ..., n

$|S| = n!$
$E_i =$ event that person $i$ gets own hat: $\pi(i) = i$

Counting single events:
$|E_i| = (n-1)!$ for all $i$

Counting pairs:
$E_iE_j : \pi(i) = i \ & \ \pi(j) = j$
$|E_iE_j| = (n-2)!$ for all $i, j$

hats: events

A sample point in $E_2$ (also in $E_5$)

All points in $E_2$

All points in $E_2 \cap E_5$
n persons at a party throw hats in middle, select at random. What is \( \Pr(\text{no one gets own hat}) \)?

\[ E_i = \text{event that person } i \text{ gets own hat} \]

\[ \Pr(\bigcup_{i=1}^{n} E_i) = \sum_i P(E_i) - \sum_{i<j} \Pr(E_i E_j) + \sum_{i<j<k} \Pr(E_i E_j E_k) \ldots \]

\[ \Pr(\text{k fixed people get own back}) = \frac{(n-k)!}{n!} \]

\[ \binom{n}{k} \text{ times that} = \frac{n!}{k!(n-k)!} \frac{(n-k)!}{n!} = \frac{1}{k!} \]

\[ \Pr(\text{none get own}) = 1 - \Pr(\text{some do}) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \ldots + (-1)^n/n! \approx \frac{1}{e} \approx .37 \]
Pr(none get own) = 1 - Pr(some do) =
1 - 1 + 1/2! - 1/3! + 1/4! … + (-1)^n/n! ≈ e^{-1} ≈ .37

Oscillates forever, but quickly converges to 1/e