Example: homeworks of 3 students returned randomly

(each permutation equally likely)

X: # people who get their own hw back

Prob	Ω	X	$(X-\overset{E(X)}{\mu})^2$
$\frac{1}{6}$	123	3	4
$\frac{1}{6}$	231	0	1
$\frac{1}{6}$	312	0	1
$\frac{1}{6}$	132	1	0
$\frac{1}{6}$	213	1	0
$\frac{1}{6}$	312	1	0

$$E(X) = 3 \cdot Pr(X=3) + 0\, Pr(X=0) + 1 \cdot Pr(X=1) = 3 \cdot \tfrac{1}{6} + 0 \cdot \tfrac{1}{3} + 1 \cdot \tfrac{1}{2} = \underline{1}$$

$$= \tfrac{1}{6} X(123) + \tfrac{1}{6} X(132) + \tfrac{1}{6} X(213) + \tfrac{1}{6} X(231) + \tfrac{1}{6} X(312) + \tfrac{1}{6} X(321)$$

2 games: in both toss fair coin

Game 1:
H You pay me \$1
T I pay you \$1

Game 2:
H You pay me \$1000
T I pay you \$1000

Same expectation. Which game would you rather play?

$\boxed{\text{Variance of r.v.:}}$ measure of its deviation from its mean

$$\mathrm{Var}(X) = E\left[(X-\mu)^2\right]$$

μ: E(X) (or mean)

called σ^2

Standard deviation $\sigma(X) = \sqrt{\mathrm{Var}(X)}$

(on same "scale" as variable)

Claim: $E\left[(X-\mu)^2\right] = E(X^2) - \mu^2$

Proof:
$$E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$$
$$= E(X^2) - E(2\mu X) + E(\mu^2)$$
linearity of expectation
$$= E(X^2) - 2\mu E(X) + \mu^2$$
$$= E(X^2) - 2\mu^2 + \mu^2$$
$$= E(X^2) - \mu^2$$

compute variance for games above

& for homework example

$X \sim Bin(n,p)$ n indep coin tosses with prob p of H

X counts # H's. (or successes)

$E(X) = np$

$Var(X) = E(X^2) - \mu^2$

$$E(X^2) = E[(X_1 + X_2 + \dots + X_n)(X_1 + X_2 + \dots + X_n)]$$

$$= \sum_{i=1}^{n} \underbrace{E(X_i^2)}_{p} + \sum_i \sum_{j \neq i} \underbrace{E(X_i X_j)}_{}$$ linearity of expectation

$$= Pr(X_i = 1, X_j = 1)$$

$$\underset{\text{indep}}{=} Pr(X_i = 1) Pr(X_j = 1)$$

$$= p^2$$

$$= np + n(n-1)p^2$$

$$Var(X) = E(X^2) - \mu^2 = np + n(n-1)p^2 - n^2p^2$$

$$= np(1-p)$$ (for which p is it maximized?)

DNA sequence each position indep A, G, T, C

with prob $p_A \quad p_G \quad p_T \quad p_C$

length = 10,000,000

Expected # of occurrences of AATGAAT ?

AAATGAATGAATCC