CDF $F_X(x) = Pr(X \le x)$ $\nearrow$ from 0 to 1

pdf $f_X(x) = \frac{d}{dx} F_X(x)$ nonnegative $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\downarrow$

$F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz$

$Pr(X \in A) = \int_A f(z)\,dz$

$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

Joint Distn's

Joint CDF: $F_{X,Y}(x,y) = Pr(X \le x, Y \le y)$

$$F(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} \underbrace{f(x,y)}_{\text{joint density fn}}\,dy\,dx$$

$$f(a,b) = \frac{\partial^2}{\partial a\,\partial b} F(a,b)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$Pr(a < X < a+da, b \le Y \le b+db) =$

$$\int_a^{a+da} \int_b^{b+db} f(x,y)\,dy\,dx \approx f(a,b)\,da\,db$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

Independence

Indep r.v.'s $\equiv$ $f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \quad \forall x,y$

$\boxed{\text{Law of Total Prob}}$

$A_1, \dots, A_n$ disjoint events that form partition of sample space

$Pr(A_i > 0) \quad \forall i$

$$f_X(x) = \sum_{i=1}^{n} Pr(A_i)\, f_{X|A_i}(x) \qquad f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{Pr(A)} & x \in A \\ 0 & \text{o.w.} \end{cases}$$

$$\Pr(E) = \int_{-\infty}^{\infty} \Pr(E \mid X = x)\, f_X(x)\, dx$$

Law of Total Expectation

$$E(X) = \sum_{i=1}^{n} \Pr(A_i)\, E(X \mid A_i) \qquad E(X \mid A) = \int_{-\infty}^{\infty} x\, f_{X \mid A}(x)\, dx$$

$$E(X) = \int_{-\infty}^{\infty} E(X \mid Y = y)\, f_Y(y)\, dy \qquad E(X \mid Y = y) = \int_{-\infty}^{\infty} x\, \underbrace{f_{X \mid Y = y}}_{\frac{f_{X,Y}(x,y)}{f_Y(y)}}(x)\, dx$$

Recipe for finding pdf of $g(X)$

$$Y = g(X)$$

1) Find CDF

$$F_Y(y) = \Pr(g(X) \le y) = \int_{\{x \mid g(x) \le y\}} f_X(x)\,dx$$

2) Differentiate to get pdf

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$X \sim U[0,1]$ What is pdf of $Y = e^X$?

Note $X > 0$ in $[0,1]$ $\Rightarrow Y > 0$ in $[1,e]$

$$\Pr(Y \le y) = \Pr(e^X \le y)$$
$$= \Pr(X \le \ln(y))$$
$$= \int_0^{\ln y} f(x)\,dx = \ln(y)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{y} \qquad 1 \le y \le e$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_1^e y \cdot \frac{1}{y}\,dy = e - 1$$

$$E(Y) = \int_{-\infty}^{\infty} e^x f_X(x)\,dx = \int_0^1 e^x\,dx = e - 1$$

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

$X_1, \dots, X_n$ iid. $U[0,1]$ What is $E(\max(X_1, \dots, X_n))$?

Let $X = \max(X_1, \dots, X_n)$

$F_X(x) = \Pr(\max(X_1, \dots, X_n) \le x) = \Pr(X_1 \le x, X_2 \le x, \dots, X_n \le x)$

$= x^n$

$f_X(x) = \frac{d}{dx} F_X(x) = n x^{n-1}$

$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x \cdot n x^{n-1} dx = n \int_0^1 x^n dx = n \frac{x^{n+1}}{n+1} \Big|_0^1 = \frac{n}{n+1}$

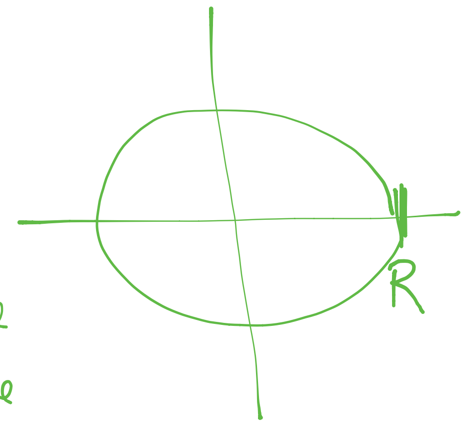
circle of radius R

centered at origin

(X,Y) coordinates of random pt in circle

("dart" equally likely to fall anywhere

$\Rightarrow f(x,y) = \begin{cases} c & x^2+y^2 \le R \\ 0 & \text{o.w.} \end{cases}$



① What is c?

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$$

$$c \iint_{x^2+y^2 \le R} dx\, dy = 1$$

use polar coordinates or observe $\iint$ = area of circle

$$\Rightarrow c = \frac{1}{\pi R^2}$$

② What is marginal density of X?

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{\pi R^2} \int_{x^2+y^2 \le R} dy$$

$$= \frac{1}{\pi R^2} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = \frac{2}{\pi R^2}\sqrt{R^2-x^2} \qquad x^2 \le R^2$$

$$0 \quad \text{o.w.}$$

③ Pr(distance from origin to pt selected ≤ a) $\qquad D = \sqrt{X^2+Y^2}$

$$F_D(a) = Pr\left(\sqrt{X^2+Y^2} \le a^2\right)$$

$$= Pr\left(X^2+Y^2 \le a^2\right)$$

$$= \int_{x^2+y^2 \le a^2} f(x,y)\,dy\,dx$$

$$= \frac{1}{\pi R^2} \underbrace{\int_{x^2+y^2 \le a^2} dy\,dx}_{\text{area of circle of radius } a} \doteq \frac{\pi a^2}{\pi R^2} = \frac{a^2}{R^2}$$

4 $E(D)$

$$f_D(a) = \frac{d}{da} F_D(a) = \frac{2a}{R^2} \quad \text{for } 0 \le a \le R$$
$$0 \quad \text{o.w.}$$

$$E(D) = \int_0^R a f_D(a) = \frac{2}{R^2}\int_0^R a^2 da = \frac{2}{3}R$$