Conditional Probability

P( | )

Conditional probability of E given F: probability that E occurs *given* that F has occurred.

"Conditioning on F"

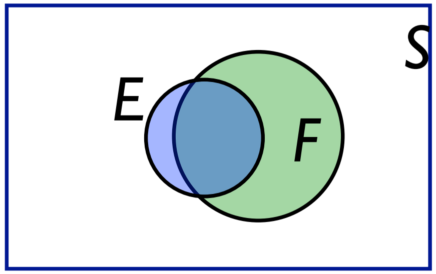
Written as P(E|F)

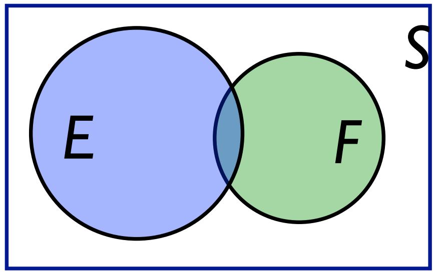
Means "P(E, given F observed)"

Sample space S reduced to those elements consistent with F (i.e. $S \cap F$)

Event space E reduced to those elements consistent with F (i.e. $E \cap F$)

With equally likely outcomes,





$$P(E \mid F) = \frac{\#\text{ of outcomes in } E \text{ consistent with } F}{\#\text{ of outcomes in } S \text{ consistent with } F} = \frac{|EF|}{|SF|} = \boxed{\frac{|EF|}{|F|}}$$

$$P(E \mid F) = \frac{|EF|}{|F|} = \frac{|EF|/|S|}{|F|/|S|} = \boxed{\frac{P(EF)}{P(F)}}$$

Suppose you flip two coins & all outcomes are equally likely.

What is the probability that both flips land on heads if…

- The first flip lands on heads?

Let B = {HH} and F = {HH, HT}

P(B|F) = P(BF)/P(F) = P({HH})/P({HH, HT})

= (1/4)/(2/4) = 1/2

- At least one of the two flips lands on heads?

Let A = {HH, HT, TH}, BA = {HH}

P(B|A) = |BA|/|A| = 1/3

- At least one of the two flips lands on tails?

Let G = {TH, HT, TT}

P(B|G) = P(BG)/P(G) = P(∅)/P(G) = 0/P(G) = 0

General defn: $P(E \mid F) = \dfrac{P(EF)}{P(F)}$ where P(F) > 0

Holds even when outcomes are *not* equally likely.

What if P(F) = 0?

P(E|F) undefined: (you can't observe the impossible)

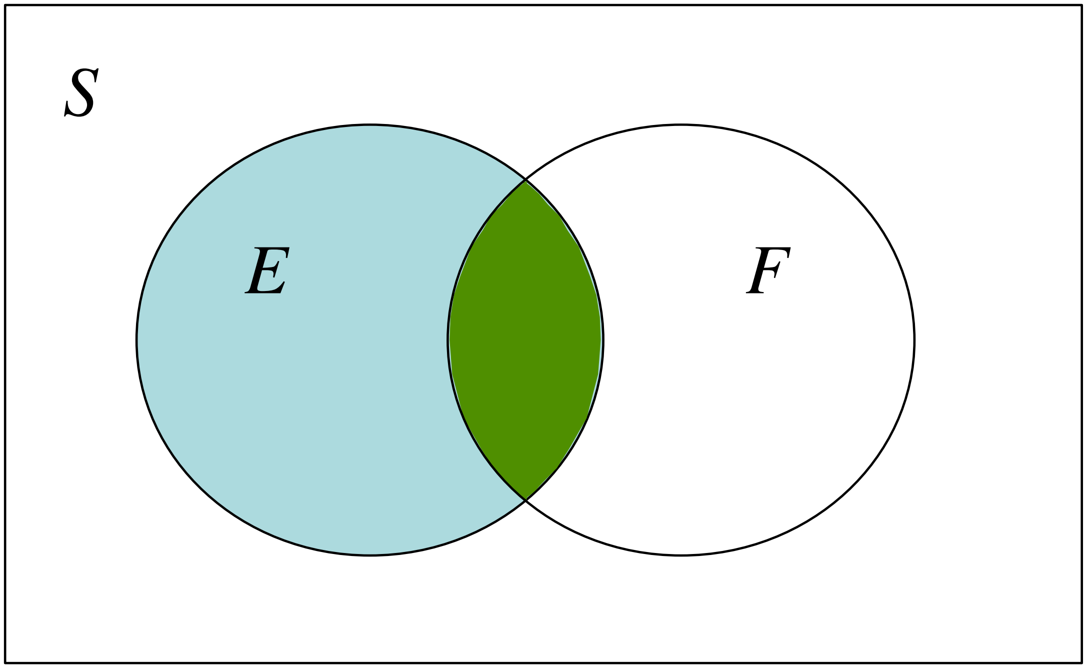
Implies: P(EF) = P(E|F) P(F) (*"the chain rule"*)

General definition of Chain Rule:

$$P(E_1 E_2 \cdots E_n) = P(E_1) P(E_2 \mid E_1) P(E_3 \mid E_1, E_2) \cdots P(E_n \mid E_1, E_2, \ldots, E_{n-1})$$

E and F are events in the sample space S

$$E = EF \cup EF^c$$



$$EF \cap EF^c = \varnothing$$

$$\Rightarrow P(E) = P(EF) + P(EF^c)$$

$P(E) = P(EF) + P(EF^c)$
$\quad = P(E|F)\ P(F) + P(E|F^c)\ P(F^c)$
$\quad = P(E|F)\ P(F) + P(E|F^c)\ (1-P(F))$

weighted average, conditioned on event F happening or not.

More generally, if $F_1, F_2, \ldots, F_n$ partition S (mutually exclusive, $\bigcup_i F_i = S$, $P(F_i)>0$), then

$P(E) = \sum_i P(E|F_i)\ P(F_i)$

weighted average, conditioned on events F_i happening or not.

(Analogous to reasoning by cases; both are very handy.)

Sally has 1 elective left to take: either Phys or Chem. She will get A with probability 3/4 in Phys, with prob 3/5 in Chem. She flips a coin to decide which to take.

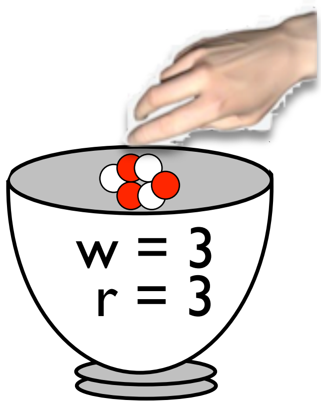
What is the probability that she gets an A?

$$P(A) = P(A|Phys)P(Phys) + P(A|Chem)P(Chem)$$
$$= (3/4)(1/2)+(3/5)(1/2)$$
$$= 27/40$$

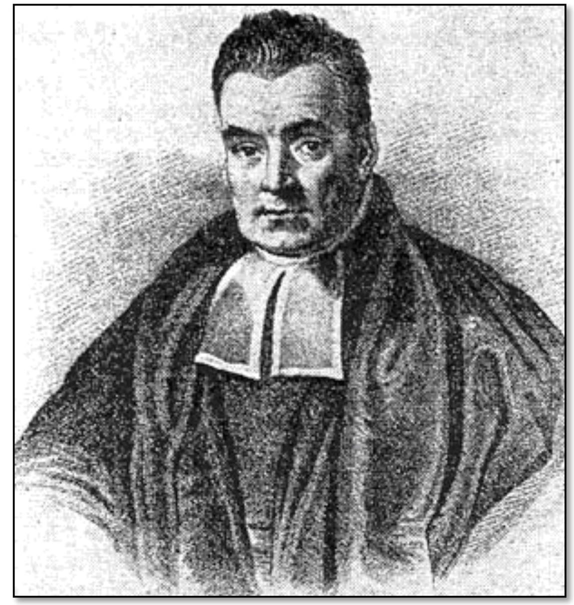
Note that conditional probability was a means to an end in this example, not the goal itself. One reason conditional probability is important is that this is a common scenario.

Bayes Theorem

6 red or white balls in an urn

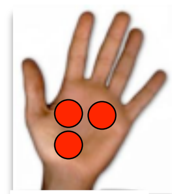


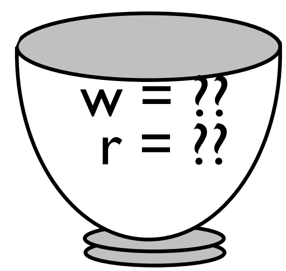
Probability of drawing 3 red balls, given 3 in urn ?



Rev. Thomas Bayes c. 1701-1761

Probability of 3 red balls in urn, given that I drew three?

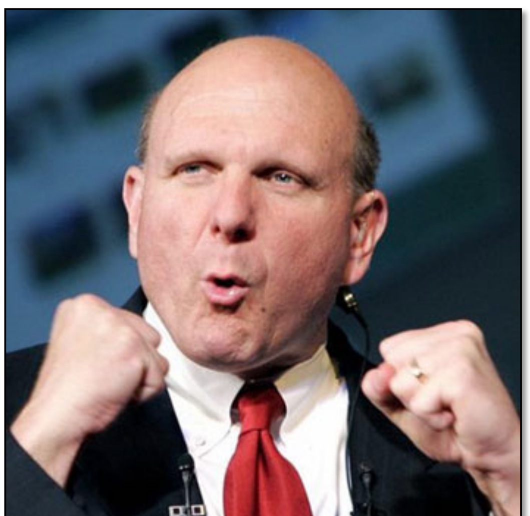




Improbable Inspiration: The future of software may lie in the obscure theories of an 18th century cleric named Thomas Bayes

Los Angeles Times (October 28, 1996)
By Leslie Helm, Times Staff Writer

When Microsoft Senior Vice President Steve Ballmer [now CEO] first heard his company was



planning a huge investment in an Internet service offering… he went to Chairman Bill Gates with his concerns…

Gates began discussing the critical role of "Bayesian" systems…

source: http://www.ar-tiste.com/latimes_oct-96.html

Most common form:

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E)}$$

Expanded form (using law of total probability):

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E \mid F)P(F) + P(E \mid F^c)P(F^c)}$$

Proof:

$$P(F \mid E) = \frac{P(EF)}{P(E)} = \frac{P(E \mid F)P(F)}{P(E)}$$

Most common form:

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E)}$$

Expanded form (using law of total probability):

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E \mid F)P(F) + P(E \mid F^c)P(F^c)}$$

Why it's important:

Reverse conditioning

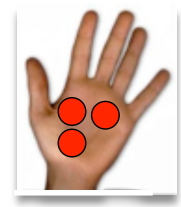
P(model | data) ~ P(data | model)

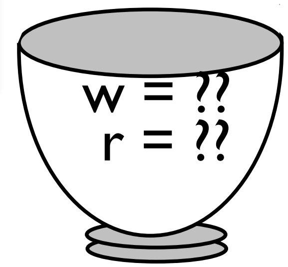
Combine new evidence (E) with prior belief (P(F))

Posterior vs prior

Bayes Theorem

An urn contains 6 balls, either 3 red + 3 white or all 6 red.
You draw 3; all are red.
Did urn have only 3 red?





Can't tell

Suppose it was 3 + 3 with probability p=3/4.
Did urn have only 3 red?

M = urn has 3 red + 3 white
D = I drew 3 red

$P(M \mid D) = P(D \mid M)P(M)/[P(D \mid M)P(M)+ P(D \mid M^c)P(M^c)]$

$P(D \mid M) = (3 \text{ choose } 3)/(6 \text{ choose } 3) = 1/20$

$P(M \mid D) = (1/20)(3/4)/[(1/20)(3/4) + (1)(1/4)] = 3/23$

prior = 3/4 ; *posterior* = 3/23

Suppose an HIV test is 98% effective in detecting HIV, i.e., its "false negative" rate = 2%. Suppose furthermore, the test's "false positive" rate = 1%.

0.5% of population has HIV

Let E = you test positive for HIV

Let F = you actually have HIV

What is P(F|E) ?

Solution:

$$
\begin{aligned}
P(F \mid E) &= \frac{P(E \mid F)P(F)}{P(E \mid F)P(F) + P(E \mid F^c)P(F^c)} \\
&= \frac{(0.98)(0.005)}{(0.98)(0.005) + (0.01)(1 - 0.005)} \\
&\approx 0.330
\end{aligned}
$$

P(E) ≈ 1.5%

Note difference between conditional and joint probability: P(F|E) = 33% ; P(FE) = 0.49%

	HIV+	HIV-
Test +	0.98 = P(E|F)	0.01 = P(E|F^c)
Test -	0.02 = P(E^c|F)	0.99 = P(E^c|F^c)

Let E^C = you test **negative** for HIV
Let F = you actually have HIV
What is $P(F|E^C)$?

$$\begin{aligned} P(F \mid E^c) &= \frac{P(E^c \mid F)P(F)}{P(E^c \mid F)P(F) + P(E^c \mid F^c)P(F^c)} \\ &= \frac{(0.02)(0.005)}{(0.02)(0.005) + (0.99)(1 - 0.005)} \\ &\approx 0.0001 \end{aligned}$$