Conditional Probability & Independence

Conditional Probabilities

- **Question**: How should we modify $\mathbb{P}(E)$ if we learn that event F has occurred?
- **Definition**: the conditional probability of E given F is

$$\boxed{\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}, \quad \text{for} \quad \mathbb{P}(F) > 0}$$

Conditional probabilities are useful because:

- Often want to calculate probabilities when some partial information about the result of the probabilistic experiment is available.
- Conditional probabilities are useful for computing "regular" probabilities.

Example 1. 2 random cards are selected from a deck of cards.

(a) What is the probability that both cards are aces given that one of the cards is the ace of spaces?

(b) What is the probability that both cards are aces given that at least one of the cards is an ace?

$\Omega = \{$all unordered pairs of cards$\}$ uniform prob dist'n

(a) $$\Pr(AA \mid A\spadesuit) = \frac{\Pr(A\spadesuit \text{ and another ace})}{\Pr(A\spadesuit)} = \frac{3}{51} \approx 0.059$$

(b) $$\Pr(AA \mid \geq 1\,A) = \frac{\Pr(AA)}{\Pr(\geq 1\,A)} = \frac{\binom{4}{2}}{\underbrace{\binom{52}{2}}_{\text{all possibilities}} - \underbrace{\binom{48}{2}}_{\text{no aces}}}$$

Cond prob satisfies the usual prob axioms.

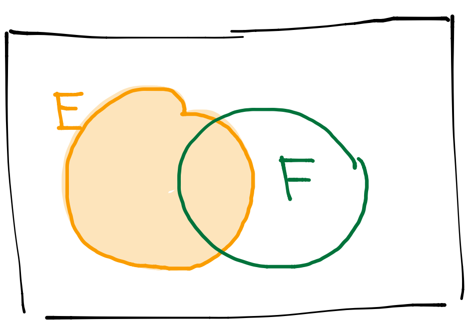
Suppose $(\mathbb{S}, \mathbb{P}(\cdot))$ is a probability space.

Then $(\mathbb{S}, \mathbb{P}(\cdot \mid F))$ is also a probability space (for $F \subset \mathbb{S}$ with $\mathbb{P}(F) > 0$).

Thus, we get all the usual identities, for example:

$\mathbb{P}(E^c \mid F) = 1 - \mathbb{P}(E \mid F)$

$\mathbb{P}(A \cup B \mid F) = \mathbb{P}(A \mid F) + \mathbb{P}(B \mid F) - \mathbb{P}(A \cap B \mid F)$



$$Pr(E|F) + Pr(E^c|F)$$

$$= \frac{Pr(E \cap F) + Pr(E^c \cap F)}{Pr(F)}$$

$$= \frac{Pr(F)}{Pr(F)} = \underline{1}$$

Cond prob satisfies the usual prob axioms.

Suppose $(\mathbb{S}, \mathbb{P}(\cdot))$ is a probability space.

Then $(\mathbb{S}, \mathbb{P}(\cdot \mid F))$ is also a probability space (for $F \subset \mathbb{S}$ with $\mathbb{P}(F) > 0$).

- $0 \leq \mathbb{P}(\omega \mid F) \leq 1$
- $\sum_{\omega \in \mathbb{S}} \mathbb{P}(\omega \mid F) = 1$
- If $E_1, E_2, \ldots, E_n$ are disjoint, then

$$\mathbb{P}(\cup_{i=1}^{n} E_i \mid F) = \sum_{i=1}^{n} \mathbb{P}(E_n \mid F)$$

Thus all our previous propositions for probabilities give analogous results for conditional probabilities.

The Multiplication Rule

- Re-arranging the conditional probability formula gives

$$\boxed{\mathbb{P}(E \cap F) = \mathbb{P}(F)\,\mathbb{P}(E \mid F)}$$

This is often useful in computing the probability of the intersection of events.

Example. Draw 2 balls at random without replacement from an urn with 8 red balls and 4 white balls. Find the chance that both are red.

8 R
4 W

draw 2 balls without replacement

$$\Pr(\text{both } R) = \Pr(\text{first } R)\,\Pr(2^{\text{nd}}\ R \mid 1^{\text{st}} \text{ red})$$

$$= \frac{4}{12} \cdot \frac{7}{11}$$

The General Multiplication Rule

$$\boxed{\begin{aligned}&\mathbb{P}(E_1 \cap E_2 \cap \cdots \cap E_n) = \\ &\qquad \mathbb{P}(E_1) \times \mathbb{P}(E_2 \,|\, E_1) \times \mathbb{P}(E_3 \,|\, E_1 \cap E_2) \times \\ &\qquad\qquad \cdots \times \mathbb{P}(E_n \,|\, E_1 \cap E_2 \cap \cdots \cap E_{n-1})\end{aligned}}$$

Example 1. I have n keys, one of which opens a lock. Trying keys at random without replacement, find the chance that the kth try opens the lock.

E_i event that i^{th} key opens door

$$Pr(E_k) = Pr(\bar{E}_1 \bar{E}_2 \cdots \bar{E}_{k-1} E_k)$$

$$= Pr(\bar{E}_1) Pr(\bar{E}_2 | \bar{E}_1) Pr(\bar{E}_3 | \bar{E}_1 \bar{E}_2) \cdots Pr(\bar{E}_{k-1} | \bar{E}_1 \cdots \bar{E}_{k-2}) Pr(E_k | \bar{E}_1 \cdots \bar{E}_{k-1})$$

$$= \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right) \cdots \left(\frac{n-(k-1)}{n-k+2}\right)\left(\frac{1}{n-k+1}\right) = \frac{1}{n}$$

duh!

The Law of Total Probability

- We know that $\mathbb{P}(E) = \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c)$.
  Using the definition of conditional probability,
  $\boxed{\mathbb{P}(E) = \mathbb{P}(E \mid F)\,\mathbb{P}(F) + \mathbb{P}(E \mid F^c)\,\mathbb{P}(F^c)}$
- This is **extremely useful**. It may be difficult to compute $\mathbb{P}(E)$ directly, but easy to compute it once we know whether or not F has occurred.

The Law of Total Probability

- To generalize, say events $F_1, \ldots, F_n$ form a **partition** if they are disjoint and $\bigcup_{i=1}^n F_i = \mathbb{S}$.
- Since $E \cap F_1, E \cap F_2, \ldots E \cap F_n$ are a disjoint partition of E

  $\boxed{\mathbb{P}(E) = \sum_{i=1}^n \mathbb{P}(E \cap F_i)}$

- Apply conditional probability to give the **law of total probability**,

  $\boxed{\mathbb{P}(E) = \sum_{i=1}^n \mathbb{P}(E \,|\, F_i)\, \mathbb{P}(F_i)}$

Bayes Formula

- Sometimes $\mathbb{P}(E \mid F)$ may be specified and we would like to find $\mathbb{P}(F \mid E)$.
- A simple manipulation gives **Bayes' formula**,

$$\boxed{\mathbb{P}(F \mid E) = \frac{\mathbb{P}(E \mid F)\, \mathbb{P}(F)}{\mathbb{P}(E)}}$$

- Combining this with the law of total probability,

$$\boxed{\mathbb{P}(F \mid E) = \frac{\mathbb{P}(E \mid F)\, \mathbb{P}(F)}{\mathbb{P}(E \mid F)\, \mathbb{P}(F) + \mathbb{P}(E \mid F^c)\, \mathbb{P}(F^c)}}$$

Bayes Formula.

$$\boxed{\mathbb{P}(F \mid E) = \frac{\mathbb{P}(E \mid F)\,\mathbb{P}(F)}{\mathbb{P}(E \mid F)\,\mathbb{P}(F) + \mathbb{P}(E \mid F^c)\,\mathbb{P}(F^c)}}$$

Example . Eric's girlfriend comes round on a given evening with probability 0.4. If she does not come round, the chance Eric watches *The Wire* is 0.8. If she does, this chance drops to 0.3. I call Eric and he says he is watching *The Wire*. What is the chance his girlfriend is around?

$$\Pr(GF \mid \text{watching}) = \frac{\Pr(\text{watching} \mid GF)\Pr(GF)}{\Pr(w \mid GF)\Pr(GF) + \Pr(w \mid \overline{GF})\Pr(\overline{GF})} = \frac{0.3 \cdot 0.4}{0.3 \cdot 0.4 + 0.8 \cdot 0.6}$$

- Sometimes conditional probability calculations can give quite unintuitive results.

Example 3. I have three cards. One is red on both sides, another is red on one side and black on the other, the third is black on both sides. I shuffle the cards and put one on the table, so you can see that the upper side is red. What is the chance that the other side is black?

- is it $1/2$, or $> 1/2$ or $< 1/2$?

Solution

Model 1: pick random card

put Red side up if there is one

$$Pr(RB \mid see\ R) = \frac{Pr(RB \cap see\ R)}{Pr(see\ R)} = \frac{1/3}{2/3} = \frac{1}{2}$$

Model 2: pick random card

pick random side to show

$$Pr(RB \mid see\ R) = \frac{Pr(RB \cap see\ R)}{Pr(see\ R)} = \frac{Pr(see\ R \mid RB)Pr(RB)}{Pr(see\ R \mid RB)\,Pr(RB) + Pr(see\ R \mid RR)\,Pr(RR) + Pr(see\ R \mid BB)\,Pr(BB)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3} + 1\cdot\frac{1}{3}} = \frac{1}{3}$$

Discussion problem. Suppose 99% of people with HIV test positive, 95% of people without HIV test negative, and 0.1% of people have HIV. What is the chance that someone testing positive has HIV?

$$\Pr(HIV^+ | test^+) = \frac{\Pr(test^+ | HIV^+)\Pr(HIV^+)}{\Pr(test^+ | HIV^+)\Pr(HIV^+) + \Pr(test^+ | HIV^-)\Pr(HIV^-)}$$

$$= \frac{.99 \cdot .001}{.99 \cdot .001 + .05 \cdot .999} = 0.019 \approx 2\%$$

If people being tested have 10% chance of being HIV^+

$\approx 68\%$

50% chance

$\approx 95\%$

Example:

Alice and Bob play a game where A tosses a coin, and wins \$1 if it lands on H or loses \$1 on T. B is surprised to find that he loses the first ten times they play. If B's **prior belief** is that the chance of A having a two headed coin is 0.01, what is his **posterior belief**?

Note. Prior and posterior beliefs are assessments of probability before and after seeing an outcome. The outcome is called **data** or **evidence**.

Solution.

$$\Pr(\text{2-headed} \mid \text{loses } 10\times) = \frac{0.01}{0.01 + \left(\frac{1}{2}\right)^{10} \cdot 0.99}$$

Independence

- Intuitively, E is independent of F if the chance of E occurring is not affected by whether F occurs. Formally,

$$\mathbb{P}(E \mid F) = \mathbb{P}(E) \tag{1}$$

- We say that E and F are **independent** if

$$\boxed{\mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F)} \tag{2}$$

Note. (2) and (1) are equivalent. when $Pr(F) > 0$
(1) not defined for $Pr(F) = 0$

Independence

$$\boxed{\mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F)} \qquad (2)$$

Note 1. It is clear from (2) that independence is a symmetric relationship. Also, (2) is properly defined when $\mathbb{P}(F) = 0$.

Note 2. (1) gives a useful way to think about independence; (2) is usually better to do the math.

Note: We can study a probabilistic model and determine if certain events are independent or we can define our probabilistic model via independence.

Example: Supposed a biased coin comes up heads with probability p, independent of other flips.

$\mathbb{P}(n \text{ heads in } n \text{ flips}) = p^n$

$\mathbb{P}(n \text{ tails in } n \text{ flips}) = (1-p)^n$

$\mathbb{P}(\text{HHTHTTT}) = p^2(1-p)p(1-p)^3 = p^{\sharp\text{H}}(1-p)^{\sharp\text{T}}$

$\mathbb{P}(\text{exactly } k \text{ heads in } n \text{ flips}) = \binom{n}{k}p^k(1-p)^{n-k}$

or non-independence

Example 1: Independence can be obvious sometimes

Draw a card from a shuffled deck of 52 cards. Let E = card is a spade and F = card is [redacted] Are E and F independent?

a heart

Solution

No complementary events!

$\Pr(E \cap F) = 0$ but $\Pr(E) = \Pr(F) = \frac{1}{4}$

Example 2: Independence can be surprising

Toss a coin 3 times. Define

A={at most one T}=$\{HHH, HHT, HTH, THH\}$

B={both H and T occur}=$\{HHH, TTT\}^c$.

Are A and B independent?

Solution

$\Pr(A) = \frac{4}{8} = \frac{1}{2}$

$\Pr(B) = \frac{6}{8} = \frac{3}{4}$

$\Pr(A \cap B) = \Pr(\{HHT, HTH, THH\})$

$= \frac{3}{8} = \Pr(A)\Pr(B)$

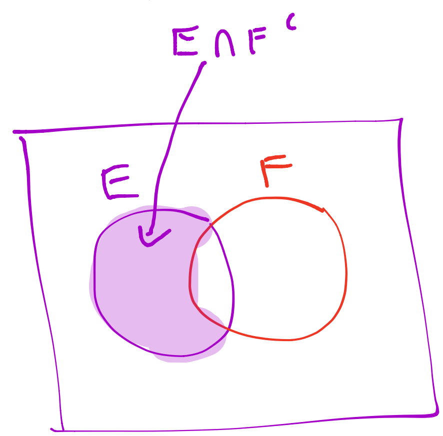
Proposition. If E and F are independent, then so are E and F^c.

Proof.

Assume E & F indep $\Leftrightarrow \Pr(E \cap F) = \Pr(E)\Pr(F)$

$$\Pr(E \cap F^c) = \Pr(E) - \Pr(E \cap F)$$
$$= \Pr(E) - \Pr(E)\Pr(F)$$
$$= \Pr(E)\left[1 - \Pr(F)\right]$$
$$= \Pr(E)\Pr(F^c)$$

$\Rightarrow$ E & F^c are indep



Independence as an Assumption

- It is often convenient to suppose independence. People sometimes assume it without noticing.

Example. A sky diver has two chutes. Let

$$E = \{\text{main chute opens}\}, \qquad \mathbb{P}(E) = 0.98;$$

$$F = \{\text{backup opens}\}, \qquad \mathbb{P}(F) = 0.90.$$

Find the chance that at least one opens, **making any necessary assumption clear**.

Note. Assuming independence does not justify the assumption! Both chutes could fail because of the same rare event, such as freezing rain.

$$1 - \Pr(\bar{E} \cap \bar{F}) = 1 - \underbrace{0.02 \cdot 0.1}_{\substack{\text{assuming} \\ \text{indep of } E \,\&\, F}} = 0.998$$

which implies indep of $\bar{E}$ and $\bar{F}$

Independence of Several Events

- Three events E, F, G are **independent** if

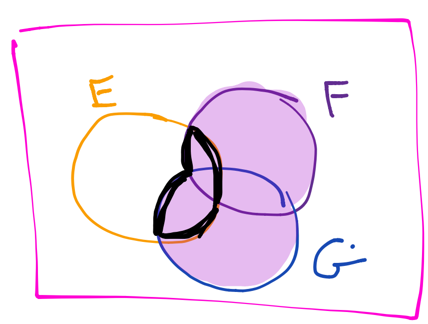
$$
\begin{aligned}
\mathbb{P}(E\cap F) &= \mathbb{P}(E)\cdot\mathbb{P}(F)\\
\mathbb{P}(F\cap G) &= \mathbb{P}(F)\cdot\mathbb{P}(G)\\
\mathbb{P}(E\cap G) &= \mathbb{P}(E)\cdot\mathbb{P}(G)\\
\mathbb{P}(E\cap F\cap G) &= \mathbb{P}(E)\cdot\mathbb{P}(F)\cdot\mathbb{P}(G)
\end{aligned}
$$

- If E, F, G are independent, then E will be independent of any event formed from F and G.

Example. Show that E is independent of $F\cup G$.

Proof.

$$
\begin{aligned}
Pr(E\cap(F\cup G)) &= Pr(E\cap F) + P(E\cap G) - Pr(E\cap F\cap G)\\
&= Pr(E)Pr(F) + Pr(E)Pr(G) - Pr(E)Pr(F)Pr(G)\\
&= Pr(E)\left[Pr(F)+Pr(G)-Pr(F\cap G)\right]\\
&= Pr(E)\,Pr(F\cup G)
\end{aligned}
$$



Pairwise Independence

- E, F and G are **pairwise independent** if E is independent of F, F is independent of G, and E is independent of G.

Example. Toss a coin twice. Set $E = \{HH, HT\}$, $F = \{TH, HH\}$ and $G = \{HH, TT\}$.

(a) Show that E, F and G are pairwise independent.

(b) By considering $\mathbb{P}(E \cap F \cap G)$, show that E, F and G are NOT independent.

(a) $Pr(E \cap F) = \frac{1}{4} = Pr(E)Pr(F)$ etc....

(b) $Pr(E \cap F \cap G) = \frac{1}{4} \neq \left(\frac{1}{2}\right)^3$

Duel Suppose that Alice and Bob are involved in a duel. The rules of the duel are that they are to pick up their guns and shoot at each other simultaneously. If one or both are hit, then the duel is over. If both shots miss, then they repeat the process. Suppose that the results of the shots are independent and that each shot of Alice will hit Bob with probability p_A and each shot of Bob will hit Alice with probability p_B. What is

(a) • the probability that the duel ends after the n^{th} round of shots?

(b) • the probability that both duelists are hit?

(c) • the conditional probability that the duel ends after the n^{th} round of shots given that both duelists are hit?

(a) $\left[\underbrace{(1-p_A)(1-p_B)}_{p}\right]^{n-1}\left[1-(1-p_A)(1-p_B)\right]$

(b) $\Pr(\text{both hit}) = \sum_{k=1}^{\infty} \Pr(\text{both hit} \cap \text{ends on } k^{th} \text{ round})$

$$= \sum_{k=1}^{\infty} p^{k-1} p_A p_B = p_A p_B \underbrace{\sum_{k=0}^{\infty} p^k}_{\frac{1}{1-p}}$$

$$\sum_{k=0}^{\infty} p^k = S$$
$$= 1 + \sum_{k=1}^{\infty} p^k$$
$$= 1 + p\sum_{k=0}^{\infty} p^k$$
$$= 1 + pS$$
$$S(1-p) = 1 \Rightarrow S = \frac{1}{1-p}$$

(c) $\Pr(\text{ends on } n^{th} \mid \text{both hit})$

$$= \frac{\Pr(\text{ends on } n^{th} \text{ \& both hit})}{\Pr(\text{both hit})}$$

<u>Baseball card collecting</u> Suppose that you continually collect baseball cards and that there are m different types. Suppose also that each time a new card is obtained, it is a type i card with probability p_i, $1 \leq i \leq m$. Suppose that you have just collected your n^{th} card. What is the probability that this is a new type?

$$\Pr(n^{th} \text{ new}) = \sum_{i=1}^{m} \underbrace{\Pr(\text{new} \mid \text{type } i)}_{(1-p_i)^{n-1}} \underbrace{\Pr(\text{type } i)}_{p_i}$$