Learning From Data: MLE

Maximum Likelihood Estimators

Parameter Estimation

Common approach in statistics: use a parametric model of data:

Assume data set:

$$Bin(n,p), \quad Poisson(\lambda), \quad N(\mu,\sigma^2)$$

$$exp(\lambda) \qquad Uniform(a,b)$$

But parameters are unknown!!! Need to estimate them.

Parameter Estimation

- Assuming sample $x_1, x_2, \ldots, x_n$ is from a parametric distribution $f(x|\theta)$, estimate θ.
- E.g.: Given sample HHTTTTTHTHTTTHH of (possibly biased) coin flips, estimate
- θ = probability of Heads

$f(x|\theta)$ is the Bernoulli probability mass function with parameter θ

Likelihood

- $P(x \mid \theta)$: Probability of event x given *model* θ
- Viewed as a function of x (fixed θ), it's a *probability*
  - E.g., $\Sigma_x P(x \mid \theta) = 1$
- Viewed as a function of θ (fixed x), it's a *likelihood*
  - E.g., $\Sigma_\theta P(x \mid \theta)$ can be anything; *relative* values of interest.

  E.g., if θ = prob of heads in a sequence of coin flips then

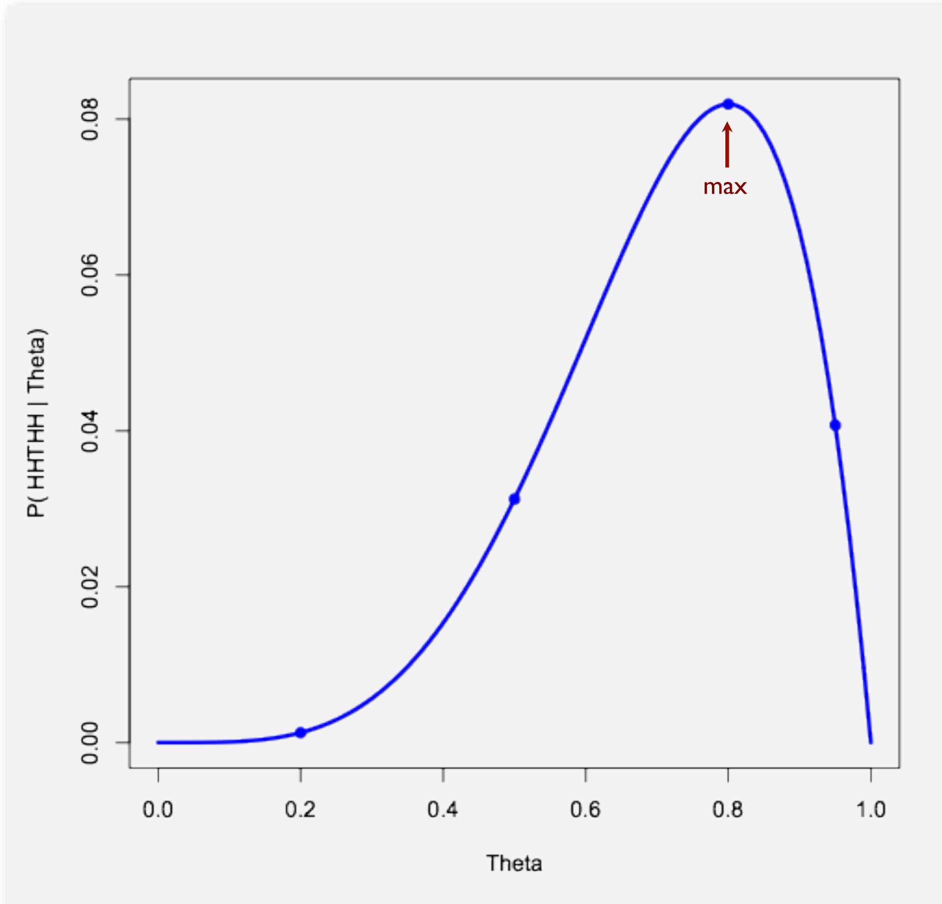
  P(HHTHH | .6) > P(HHTHH | .5),

  I.e., event HHTHH is *more likely* when θ = .6 than θ = .5
  - And what θ make HHTHH *most* likely?

Likelihood Function

- $P(\text{ HHTHH } | \ \theta\ )$: Probability of HHTHH, given $P(H) = \theta$:

θ	$\theta^4(1-\theta)$
0.2	0.0013
0.5	0.0313
0.8	0.0819
0.95	0.0407



Maximum Likelihood Parameter Estimation

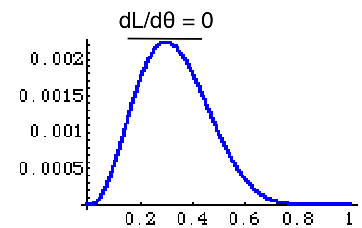
- One (of many) approaches to param. est.
- Likelihood of (indp) observations $x_1, x_2, \ldots, x_n$

$$L(x_1, x_2, \ldots, x_n \mid \theta) = \prod_{i=1}^{n} f(x_i \mid \theta)$$

- As a function of θ, what θ maximizes the likelihood of the data actually observed
- Typical approach: $\frac{\partial}{\partial\theta} L(\vec{x} \mid \theta) = 0$ or $\frac{\partial}{\partial\theta} \log L(\vec{x} \mid \theta) = 0$

Example 1

• n coin flips, $x_1, x_2, ..., x_n$; n_0 tails, n_1 heads, $n_0 + n_1 = n$;
θ = probability of heads



$$L(x_1, x_2, \ldots, x_n \mid \theta) = (1-\theta)^{n_0}\theta^{n_1}$$

$$\log L(x_1, x_2, \ldots, x_n \mid \theta) = n_0 \log(1-\theta) + n_1 \log \theta$$

$$\frac{\partial}{\partial\theta} \log L(x_1, x_2, \ldots, x_n \mid \theta) = \frac{-n_0}{1-\theta} + \frac{n_1}{\theta}$$

Setting to zero and solving:

$$\boxed{\hat{\theta} = \frac{n_1}{n}}$$

Observed fraction of successes in *sample* is MLE of success probability in *population*

(Also verify it's max, not min, & not better on boundary)

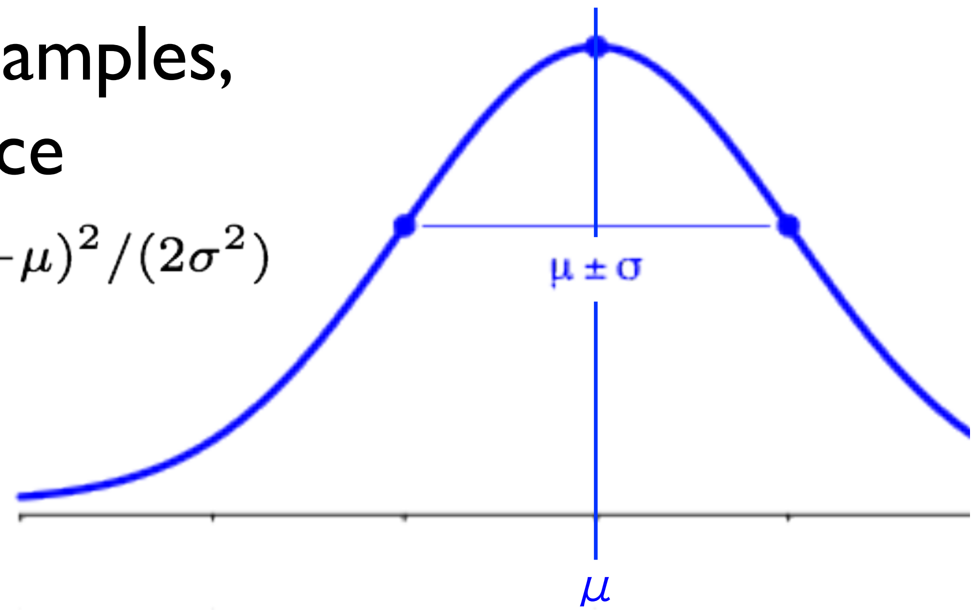
Parameter Estimation

- Assuming sample $x_1, x_2, \ldots, x_n$ is from a parametric distribution $f(x|\theta)$, estimate θ.

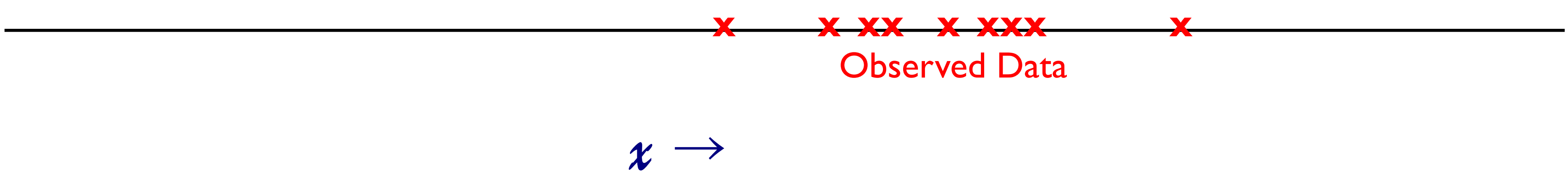
- E.g.: Given n normal samples, estimate mean & variance

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$$

$$\theta = (\mu, \sigma^2)$$

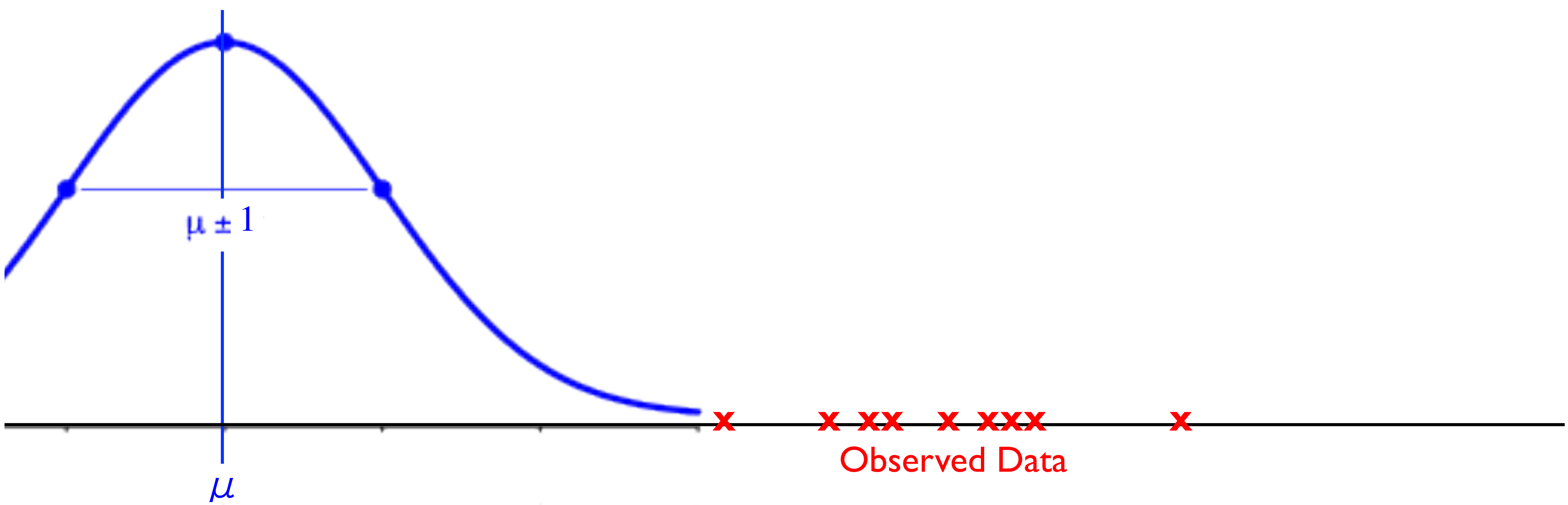


Ex2: I got data; a little birdie tells me it's normal, and promises $\sigma^2 = 1$



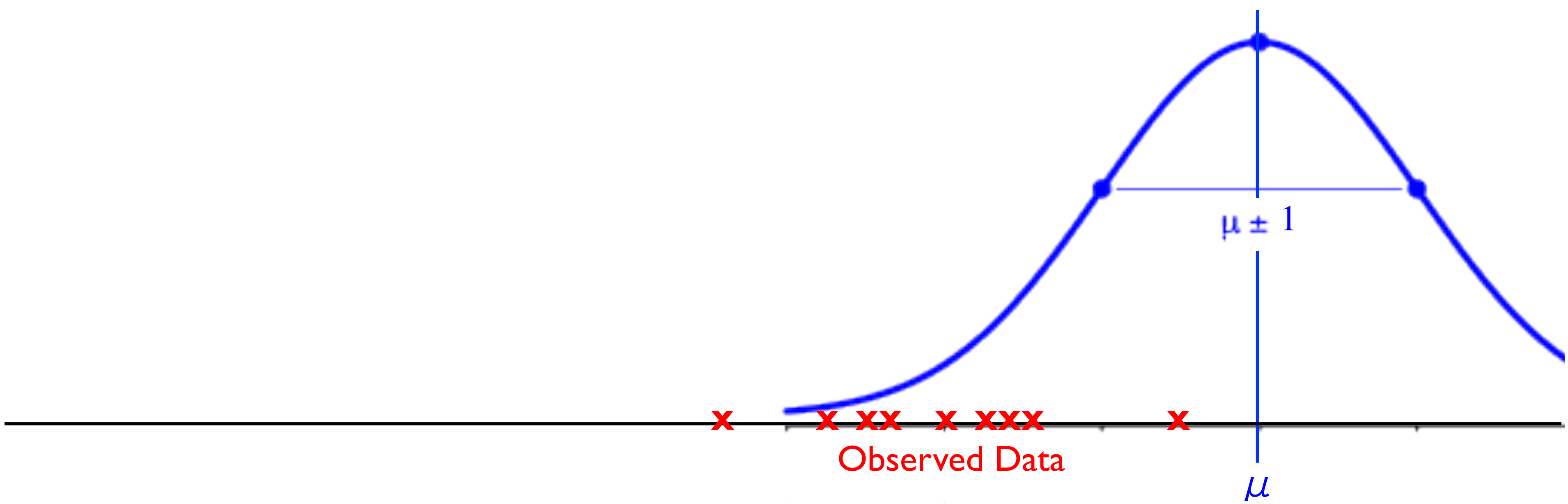
Which is more likely: (a) this?

μ unknown, $\sigma^2 = 1$



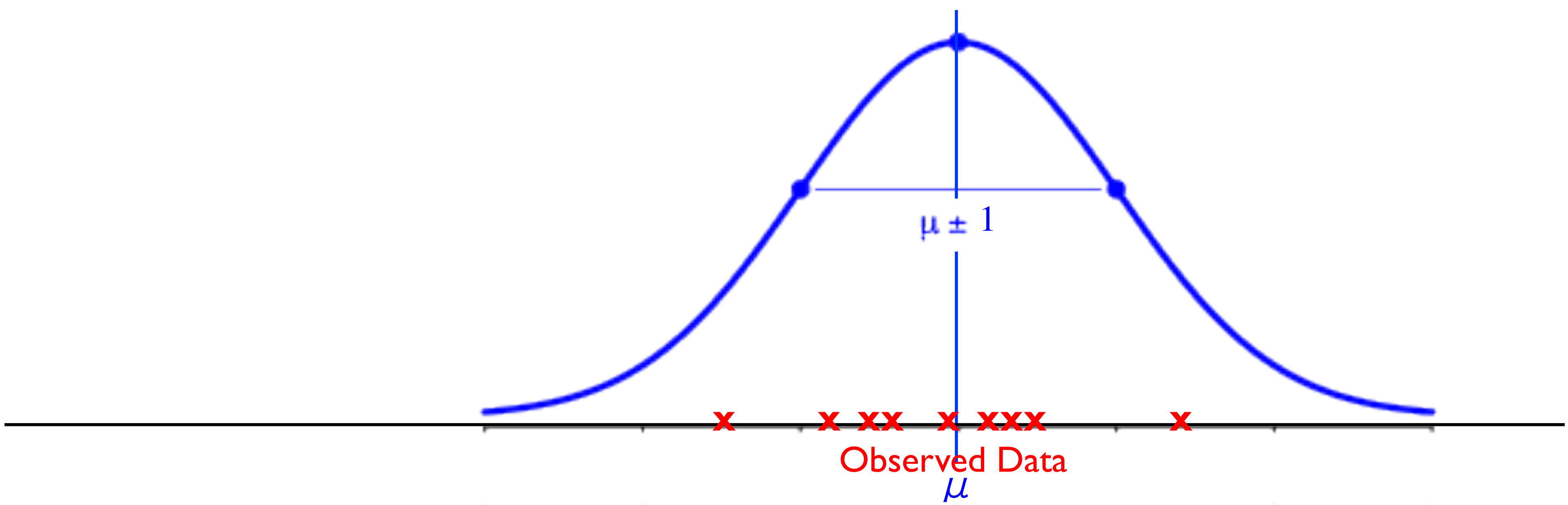
Which is more likely: (b) or this?

μ unknown, $\sigma^2 = 1$



Which is more likely: (c) or *this*?

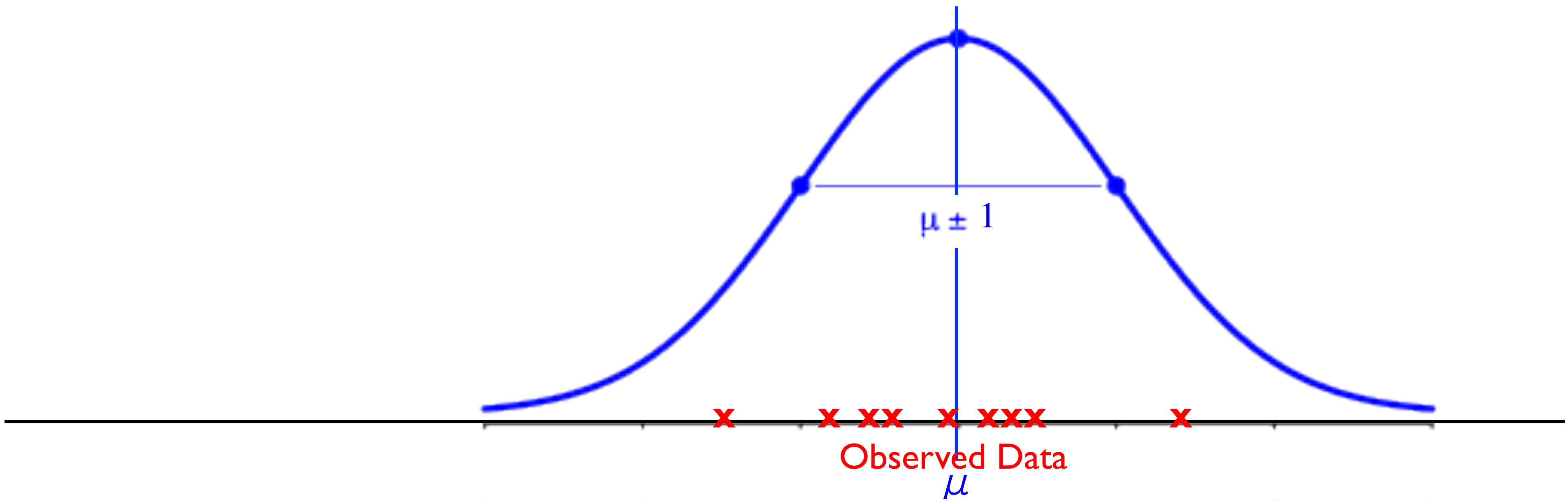
μ unknown, $\sigma^2 = 1$



Which is more likely: (c) or *this*?

μ unknown, $\sigma^2 = 1$

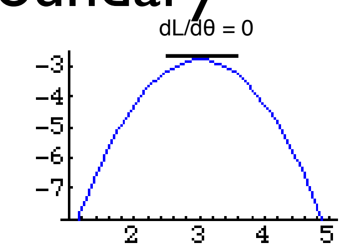
Looks good by eye, but how do I optimize my estimate of μ ?



Ex. 2: $x_i \sim N(\mu, \sigma^2),\ \sigma^2 = 1,\ \mu$ unknown

$$
\begin{aligned}
L(x_1, x_2, \ldots, x_n|\theta) &= \prod_{1 \le i \le n} \frac{1}{\sqrt{2\pi}} e^{-(x_i-\theta)^2/2} \\
\ln L(x_1, x_2, \ldots, x_n|\theta) &= \sum_{1 \le i \le n} -\frac{1}{2}\ln 2\pi - \frac{(x_i-\theta)^2}{2} \\
\frac{d}{d\theta} \ln L(x_1, x_2, \ldots, x_n|\theta) &= \sum_{1 \le i \le n} (x_i - \theta) \\
&= \left(\sum_{1 \le i \le n} x_i\right) - n\theta \;=\; 0
\end{aligned}
$$

And verify it's max, not min & not better on boundary



$$
\boxed{\hat{\theta} \;=\; \left(\sum_{1 \le i \le n} x_i\right)/n \;=\; \bar{x}}
$$

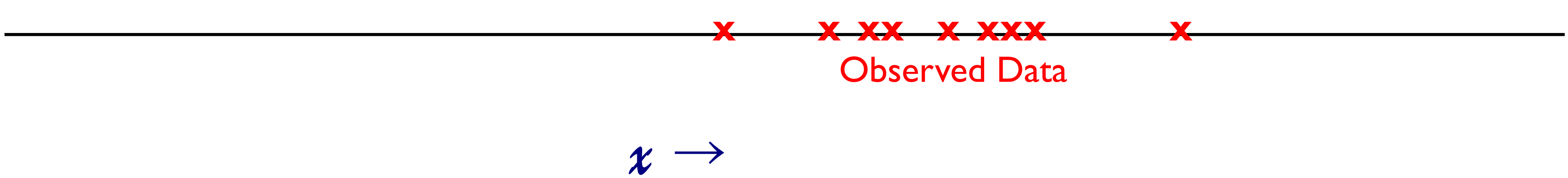
Sample mean is MLE of population mean

Hmm …, density ≠ probability

- So why is "likelihood" function equal to product of *densities*??
    - a) for maximizing likelihood, we really only care about *relative* likelihoods, and density captures that
- and/or
    - b) if density at x is $f(x)$, for any small $\delta > 0$, the probability of a sample within $\pm\delta/2$ of x is $\approx \delta f(x)$, but δ is *constant* wrt θ, so it just drops out of

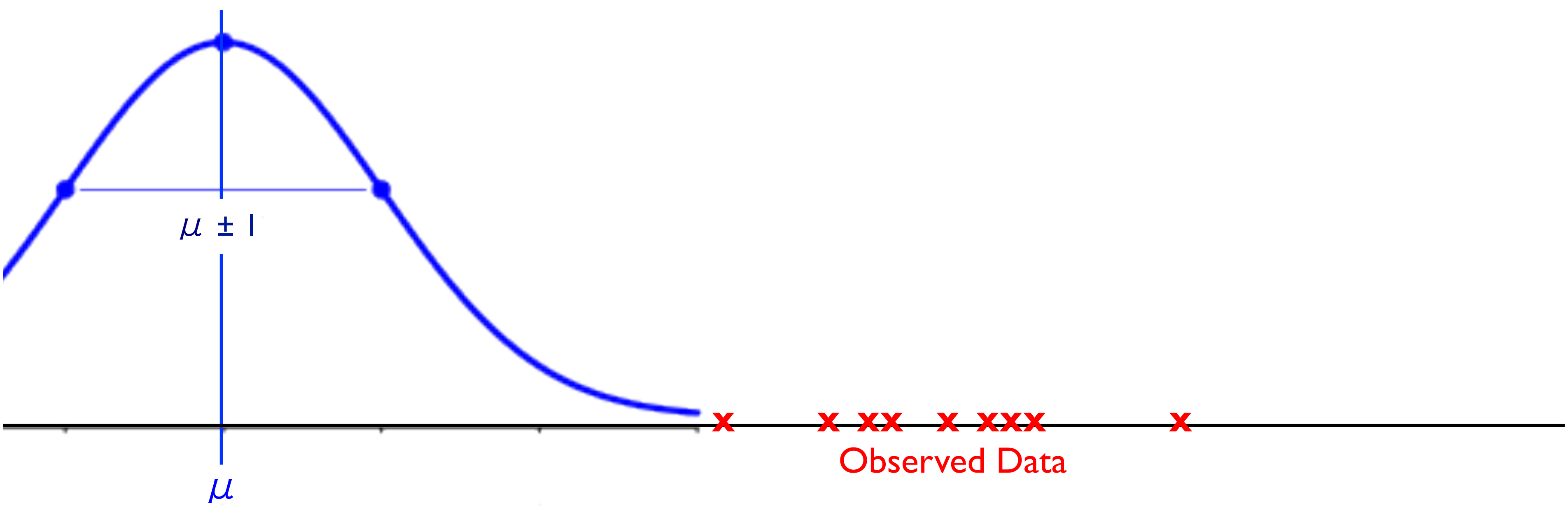
      $d/d\theta \log L(\ldots) = 0.$

Ex3: I got data; a little birdie tells me it's normal (but does *not* tell me σ^2)



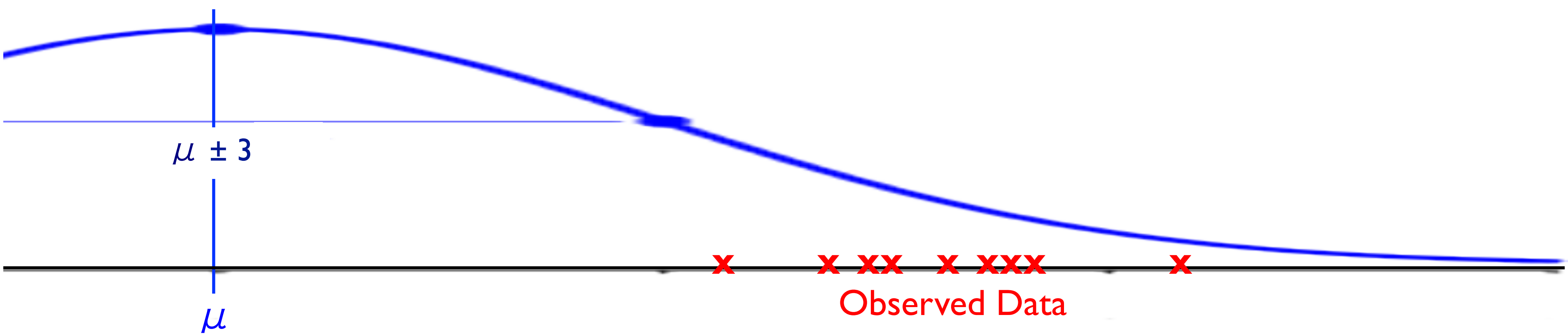
Which is more likely: (a) this?

μ, σ^2 both unknown



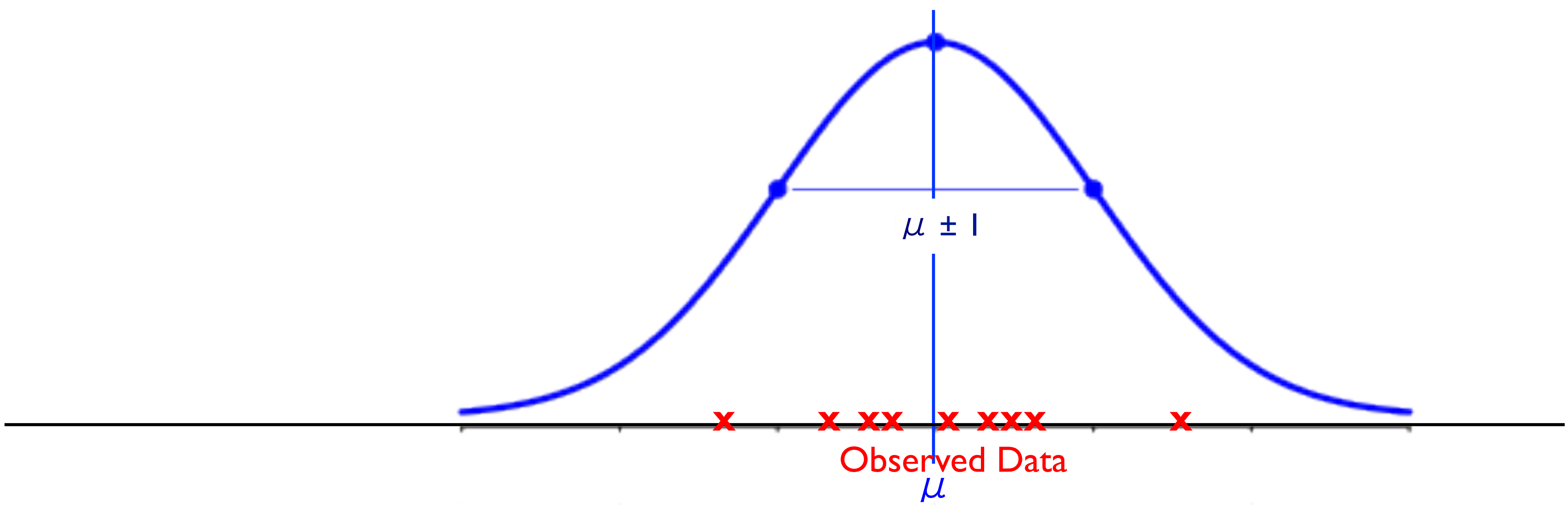
Which is more likely: (b) or this?

μ, σ^2 both unknown



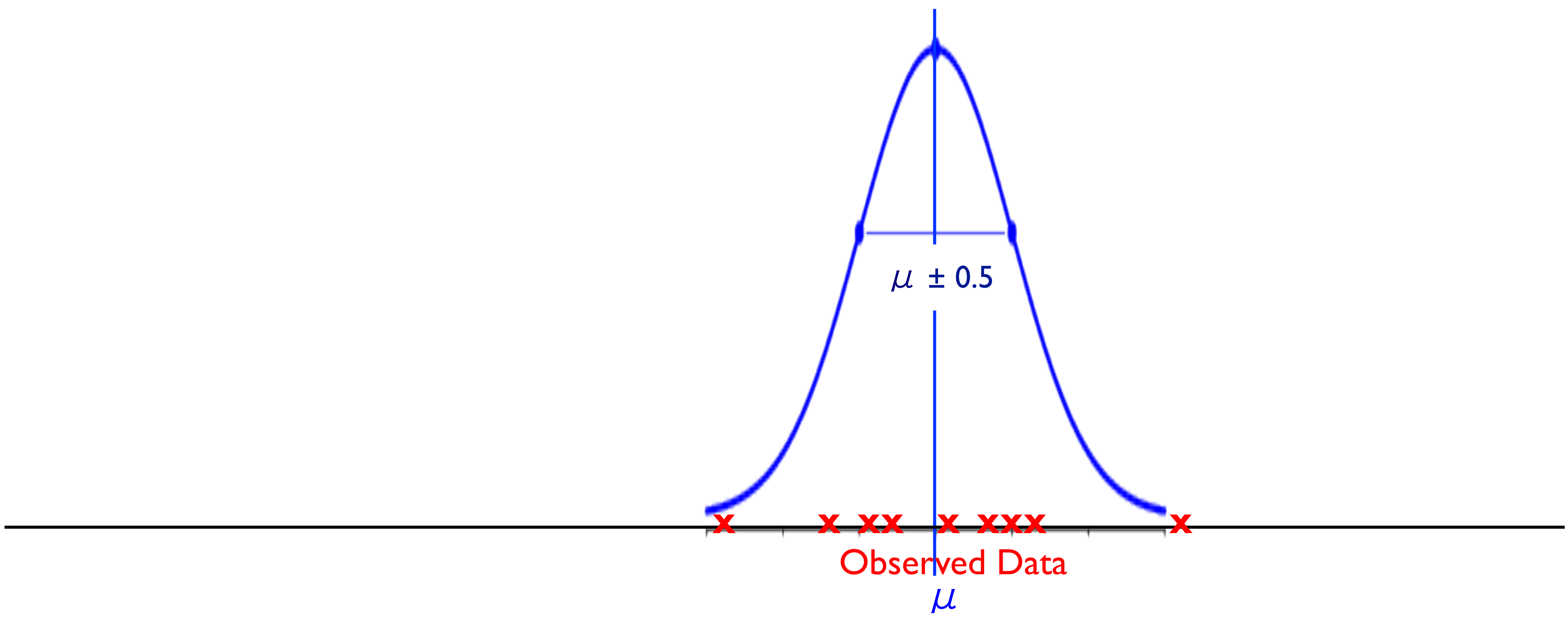
Which is more likely: (c) or this?

μ, σ^2 both unknown



Which is more likely: (d) or *this*?

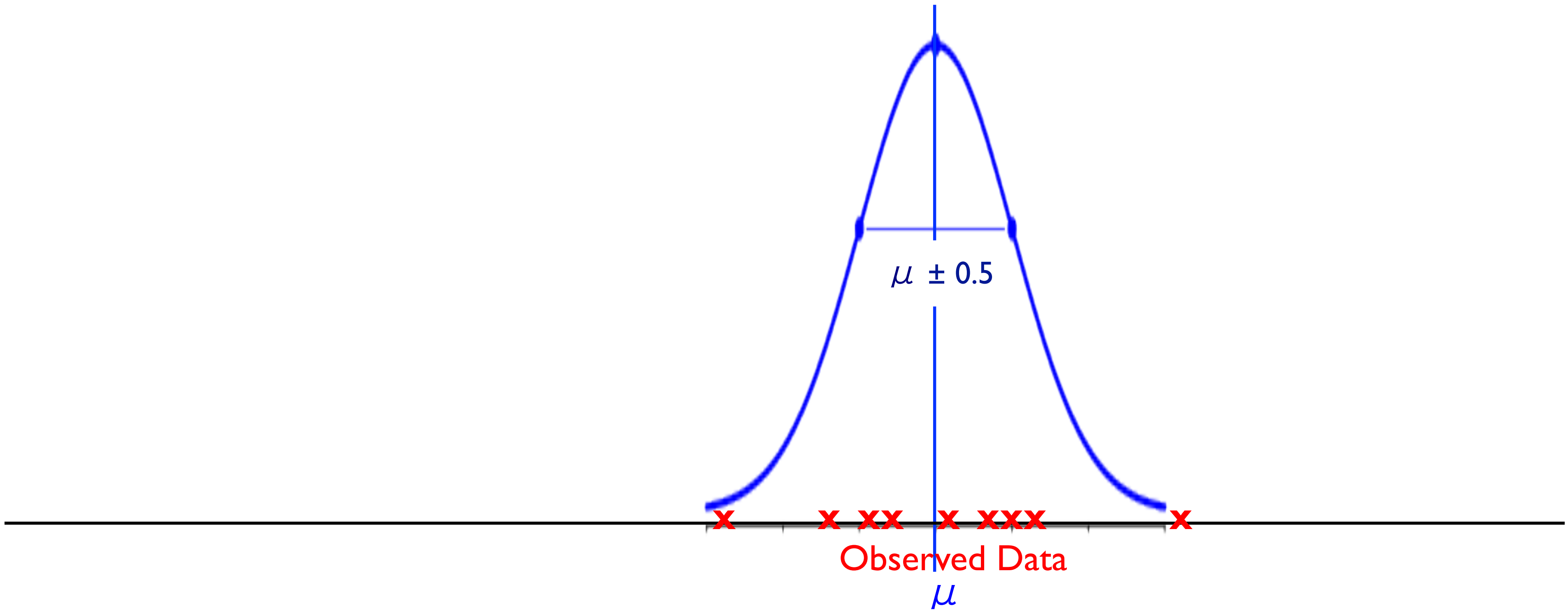
μ, σ^2 both unknown



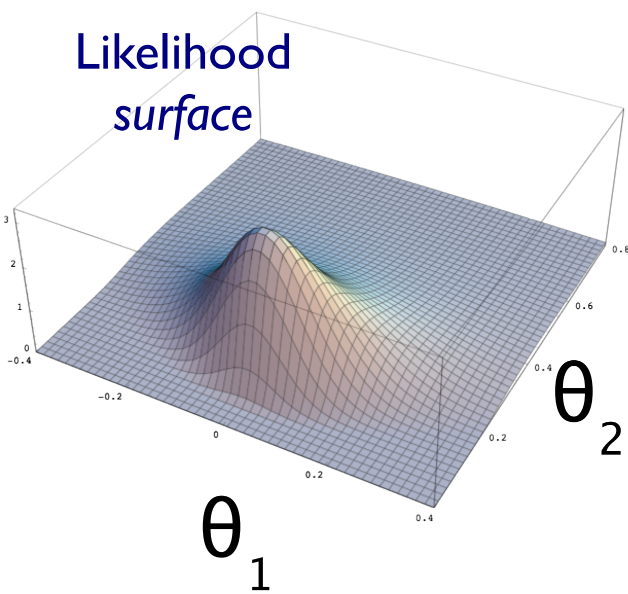
Which is more likely: (d) or *this*?

μ, σ^2 both unknown

Looks good by eye, but how do I optimize my estimates of μ & σ^2 ?



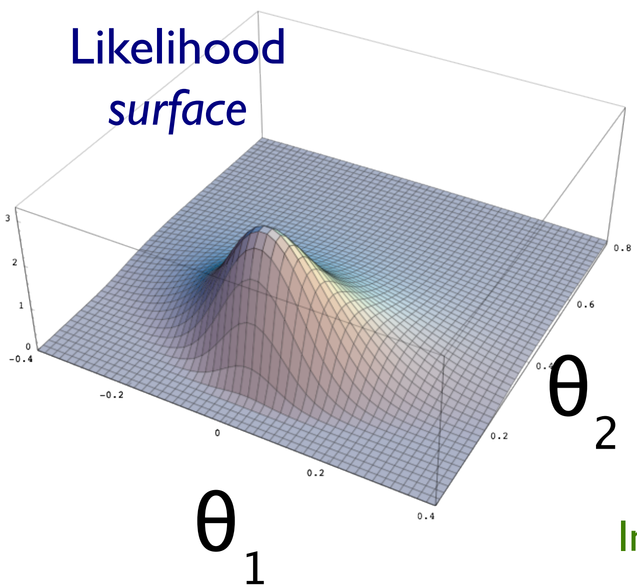
Ex 3: $x_i \sim N(\mu, \sigma^2),\ \mu, \sigma^2$ both unknown



Ex 3: $x_i \sim N(\mu, \sigma^2),\ \mu, \sigma^2$ both unknown

$$\ln L(x_1, x_2, \ldots, x_n | \theta_1, \theta_2) = \sum_{1 \le i \le n} -\frac{1}{2} \ln 2\pi\theta_2 - \frac{(x_i - \theta_1)^2}{2\theta_2}$$

$$\frac{\partial}{\partial \theta_1} \ln L(x_1, x_2, \ldots, x_n | \theta_1, \theta_2) = \sum_{1 \le i \le n} \frac{(x_i - \theta_1)}{\theta_2} = 0$$



$$\boxed{\hat{\theta}_1 = \left(\sum\nolimits_{1 \le i \le n} x_i\right) / n = \bar{x}}$$

Sample mean is MLE of population mean, again

In general, a problem like this results in 2 equations in 2 unknowns.
Easy in this case, since θ_2 drops out of the $\partial / \partial \theta_1 = 0$ equation

Ex. 3, (cont.)

$$\ln L(x_1, x_2, \ldots, x_n|\theta_1, \theta_2) = \sum_{1 \le i \le n} -\frac{1}{2} \ln 2\pi\theta_2 - \frac{(x_i - \theta_1)^2}{2\theta_2}$$

$$\frac{\partial}{\partial\theta_2} \ln L(x_1, x_2, \ldots, x_n|\theta_1, \theta_2) = \sum_{1 \le i \le n} -\frac{1}{2} \frac{2\pi}{2\pi\theta_2} + \frac{(x_i - \theta_1)^2}{2\theta_2^2} = 0$$

$$\boxed{\hat{\theta}_2 = \left(\sum_{1 \le i \le n} (x_i - \hat{\theta}_1)^2\right) / n = \bar{s}^2}$$

Sample variance is MLE of *population* variance

Summary

- MLE is *one* way to estimate *parameters* from *data*
- You choose the *form* of the model (normal, binomial, ...)
- Math chooses the *value(s)* of parameter(s)
- Has the intuitively appealing property that the parameters maximize the *likelihood* of the observed data; basically just assumes your sample is "representative"