8. Average-Case Analysis of Algorithms + Randomized Algorithms

1) Probability tools you've seen allow formal definition of "average case" running time of algorithms

2) Coupled with a few analysis tricks you'll see in more detail in 421 or elsewhere, you can analyze those algorithms, and

3) Adding randomness to *algorithms* can have surprising benefits, and again, you've got the basic tools needed to understand the issues and do the necessary analysis

4) Specifics: "average" case analysis of insertion sort and quicksort, and randomized quicksort

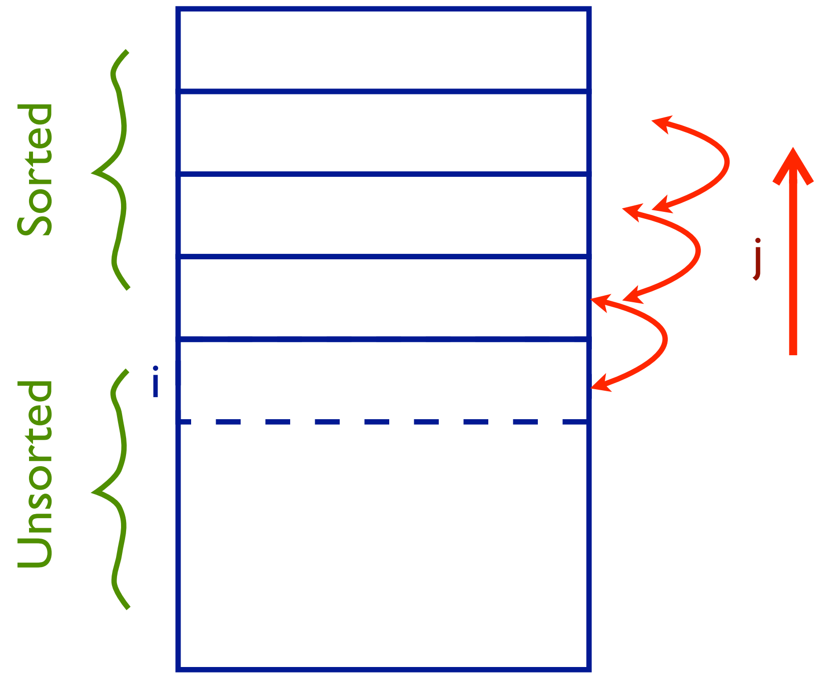
insertion sort

```
Array A[1] … A[n]
for i = 2 … n-1 {
  T = A[i]
  j = i-1
  while j >= 0 && T < A[j] {
    A[j+1] = A[j]
    A[j] = T
    j = j-1
  }
  A[j+1] = T
```

"compare"

"swap"

or



Run Time

Worst Case: $O(n^2)$

((n choose 2) swaps; #compares = #swaps + n - 1)

"Average Case"

? What's an "average" input?

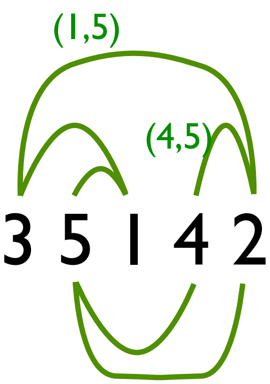
One idea (and about the only one that is analytically tractable): assume all n! permutations of input are equally likely.

A *permutation* $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ of 1, ..., n is simply a list of the numbers between 1 and n, in some order.

(i,j) is *an inversion* in π if $i < j$ but $\pi_i > \pi_j$

G. Cramer, 1750

E.g.,



$$\pi = (\ 3\ 5\ 1\ 4\ 2\ )$$

has six inversions: (1,3), (1,5), (2,3), (2,4), (2,5), and (4,5)

Min possible: 0: $\pi = (\ 1\ 2\ 3\ 4\ 5\ )$

Max possible: n choose 2: $\pi = (\ 5\ 4\ 3\ 2\ 1\ )$

Obviously, the goal of sorting is to remove inversions

Swapping an *adjacent* pair of positions that are *out-of-order* decreases the number of inversions by *exactly 1*.

So..., number of swaps performed by insertion sort is exactly the number of inversions present in the input.

Counting them:

a. worst case: n choose 2

b. average case:

$$I_{i,j} = \begin{cases} 1 & \text{if } (i,j) \text{ is an inversion} \\ 0 & \text{if not} \end{cases}$$

$$I = \sum_{i<j} I_{i,j}$$

"The method of indicators"

$$E[I] = E\left[\sum_{i<j} I_{i,j}\right] = \sum_{i<j} E\left[I_{i,j}\right]$$

There is a 1-1 correspondence between permutations *having* inversion (i,j) versus *not*:

$$\begin{array}{lccccc} & & i & & j & \\ \pi & (\cdots & a & \cdots & b & \cdots) \\ \pi' & (\cdots & b & \cdots & a & \cdots) \end{array}$$

So:

when π is chosen uniformly at random

$$E[I_{i,j}] = P(I_{i,j} = 1) = 1/2$$

$$E[I] = \sum_{i<j} E[I_{i,j}] = \sum_{i<j} \frac{1}{2} = \binom{n}{2} \cdot \frac{1}{2}$$

Thus, the expected number of swaps in insertion sort is $\binom{n}{2}/2$ versus $\binom{n}{2}$ in worst-case. I.e.,

> The average run time of insertion sort (assuming random input) is about half the worst case time.

average-case analysis of quicksort

Quicksort also does swaps, but *non*adjacent ones.

Recall method:

Array A[1..n]

1. "pivot" = A[1]
2. "Partition" (O(n) compares/swaps) so that:

   {A[1], ..., A[i-1]} < {A[i] == pivot} < {A[i+1], ..., A[n]}

3. recursively sort {A[1], ..., A[i-1]} & {A[i+1], ..., A[n]}

Worst case: already sorted (among others) –

$T(n) = n + T(n-1) \Rightarrow$

$= n + (n-1) + (n-2) + \dots + 1 = n(n+1)/2$

Best case: pivot is always median

$T(n) = 2\,T(n/2) + n$

$\Rightarrow \sim n \log_2 n$

Average case: ?

Below. Will turn out to be ~40% slower than best

Why?

Random pivots are "near the middle on average"

Assume input is a random permutation of 1, ..., n, i.e., that all n! permutations are equally likely

Then 1^{st} pivot A[1] is uniformly random in 1, ..., n

Important subtlety:

pivots at all recursive levels will be random, too, (unless you do something funky in the partition phase)

Let C_N be the average number of comparisons made by quicksort when called on an array of size N. Then:

$C_0 = C_1 = 0$ (a list of length ≤ 1 is already sorted)

In the general case, there are N-1 comparisons: the pivot vs every other element (a detail: plus 2 more for handling the "pointers cross" test to end the loop). The pivot ends up in some position $1 \leq k \leq N$, leaving two subproblems of size k-1 and N-k. By Law of Total Expectation:

$$C_N = N + 1 + \frac{1}{N} \sum_{1 \leq k \leq N} (C_{k-1} + C_{N-k}) \quad \text{for } N \geq 2,$$

$1/N$ because all values $1 \leq k \leq N$ for pivot are equally likely.

(Analysis from Sedgewick, *Algorithms in C*, 3rd ed., 1998, p311-312; Knuth TAOCP v3, 1st ed 1973, p120.)

$$C_N = N + 1 + \frac{1}{N} \sum_{1 \le k \le N} (C_{k-1} + C_{N-k}) \quad \text{for } N \ge 2,$$

Rearrange; every C_i is there twice

$$C_N = N + 1 + \frac{2}{N} \sum_{1 \le k \le N} C_{k-1}.$$

Multiply by N; subtract same for N-1

$$NC_N - (N-1)C_{N-1} = N(N+1) - (N-1)N + 2C_{N-1}.$$

Rearrange

$$NC_N = (N+1)C_{N-1} + 2N.$$

$$NC_N = (N+1)C_{N-1} + 2N.$$

div by N(N+1)

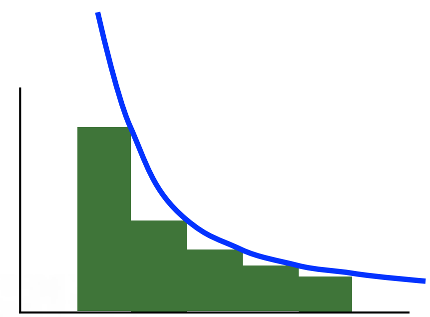
$$\frac{C_N}{N+1} = \frac{C_{N-1}}{N} + \frac{2}{N+1}$$

substitute

$$= \frac{C_{N-2}}{N-1} + \frac{2}{N} + \frac{2}{N+1}$$

$$= \vdots$$

$$= \frac{C_2}{3} + \sum_{3 \le k \le N} \frac{2}{k+1}.$$



$$\frac{C_N}{N+1} \approx 2 \sum_{1 \le k < N} \frac{1}{k} \approx 2 \int_1^N \frac{1}{x} dx = 2 \ln N,$$

$$\boxed{2N \ln N \approx 1.39 N \lg N}$$

So, *average* run time, averaging over *randomly ordered inputs,* = Θ(n log n).

A worst case input is still worst case: n^2 every time

(Is real data random?)

Is it possible to improve the worst case?

another idea: randomize the algorithm

Algorithm as before, except pivot is a *randomly selected* element of A[1]...A[n] (at top level; A[i]..A[j] for subproblem i..j)

Analysis is the same, but conclusion is different:

On *any* fixed input, average run time is n log n, *averaged over repeated (random) runs of the algorithm*.

There are no longer any "bad inputs", just "bad (random) choices." Fortunately, such choices are improbable!

Average Case Analysis (of a deterministic alg):

1. for algorithm A, choose a sample space S and probability distribution P from which inputs are drawn
2. for $x \in S$, let $T(x)$ be the time taken by A on input x
3. calculate, as a function of the "size," n, of inputs,
   $\Sigma_{x \in S} T(x) \bullet P(x)$
   which is the expected or average run time of A

For sorting, distrib is usually "all n! permutations equiprobable"

Insertion sort: E[time] ∝ E[inversions] = $\binom{n}{2}/2$ = $\Theta(n^2)$, about half the worst case

Quicksort: E[time] = $\Theta(n \log n)$ vs $\Theta(n^2)$ in worst case; fun with recurrences, sums & integrals

Randomized Algorithms (with non-random input):

1. for a randomized algorithm A, *input* x is fixed, just as usual, from some space $\mathcal{I}$ of possible inputs, but the algorithm may draw (and use) random samples y = (y_1, y_2, ...) from a given sample space S and probability distribution P. E.g., y_i = "which pivot in subproblem i"
2. for *any* x $\in \mathcal{I}$ and any y $\in$ S, let T(x,y) be the time taken by A on input x when y is sampled from S
3. calculate, as a function of the "size," n, of inputs,

   $$\max_{x \in \mathcal{I}} \sum_{y \in S} T(x,y) \cdot P(y)$$

   which is the expected or average run time of A on a worst-case input

Randomized Quicksort: choosing pivots at random, E[time] = Θ(n log n) for *any* input. (For every input, there are some rare random choice sequences causing n^2 time.)

Key distinction:

If average case analysis of a (deterministic) algorithm *D* says that average runtime ≪ worst case, then worst case inputs must be rare. *But if you get one, your bad luck is permanent: D will be slow time after time after time on that input…*

If expected run time of a randomized algorithm *R* is ≪ worst case, some inputs may be worse than others, but *there are no bad inputs*. If *R* runs slowly (near worst case) once, on a specific input, *your bad luck is transient; if you run it again you can expect it to run near the overall expectation.*

Worst-case analysis is much more common than *average-case* analysis because:

- It's often easier
- To get meaningful average case results, a reasonable probability model for "typical inputs" is critical, but may be unavailable, or difficult to analyze
- The results are often similar (e.g., insertion sort)

But in some important examples, average-case is sharply better (e.g., quicksort)

Randomized algorithms are very important in many areas; sometimes easier to argue that bad stuff is rare than to deterministically circumvent it (e.g., randomized qsort)

- Fascinating and deep open problem: is this intrinsic?