counting is hard with only 10 fingers

How many ways to do $X$?

$X =$ “Choose an integer between one and ten.”

$X =$ “Walk from 1\textsuperscript{st} and Spring to 5\textsuperscript{th} and Pine.”
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$X = \text{“Walk from 1}\text{st and Spring to 5}\text{th and Pine.”}$

Counting is hard when numbers are large or constraints are complex.

We need a systematic approach.
the basic principle of counting (product rule)

If one of $m$ outcomes is chosen from $A$, followed sequentially by one of $n$ outcomes from $B$, then there are...

$m \times n$ outcomes possible overall.

Generalizes to more events.
How many n-bit numbers are there?

$$2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$$

How many subsets of a set of size n are there?

$$\{1, 2, 3, \ldots, n\}$$

Set contains 1 or doesn’t contain 1.
Set contains 2 or doesn’t contain 2.
Set contains 3 or doesn’t contain 3...

$$2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$$
How many 4-character passwords are there if each character must be one of a, b, c, ..., z, 0, 1, 2, ..., 9?

\[36 \cdot 36 \cdot 36 \cdot 36 = 1,679,616 \approx 1.7 \text{ million}\]

Same question, but now characters cannot be repeated...

\[36 \cdot 35 \cdot 34 \cdot 33 = 1,413,720 \approx 1.4 \text{ million}\]
How many arrangements of the letters \{a,b,c\} are possible (using each once, no repeat, order matters)?

\[
\begin{array}{ccc}
  a & b & c \\
  b & a & c \\
  c & a & b \\
  a & c & b \\
  b & c & a \\
  c & b & a \\
\end{array}
\]

More generally, how many arrangements of \(n\) distinct items are possible?

\[n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n! \quad (n \text{ factorial})\]
permutations

Q. How many permutations of DOGIE are there?

\[ 5! = 120 \]

Q. How many of DOGGY?

\[ \frac{5!}{2!} = 60 \]

Q. How many of GODOGGY?

\[ \frac{7!}{3!2!1!1!1!1!} = 420 \]
Your dark elf avatar can carry three objects chosen from:

How many ways can he/she be equipped?

\[
\frac{5 \cdot 4 \cdot 3}{3!} = \frac{5!}{3! \cdot 2!} = 10
\]
Combinations: Number of ways to choose $r$ things from $n$ things

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

Pronounced “n choose r” aka “binomial coefficients”

E.g., \[\binom{n}{2} = \frac{n(n-1)}{2} = \Theta(n^2)\]

Many identities:

\[
\binom{n}{r} = \binom{n}{n-r} \quad \leftarrow \text{by symmetry of definition}
\]
\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \quad \leftarrow \text{1st object either in or out}
\]
\[
\binom{n}{r} = \frac{n(n-1)}{r(r-1)} \quad \leftarrow \text{team + captain}
\]
the binomial theorem

$$(x + y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$$

**Proof 1:** Induction …

**Proof 2:** Counting

$$(x+y) \cdot (x+y) \cdot (x+y) \cdot \ldots \cdot (x+y)$$

Pick either $x$ or $y$ from first factor
Pick either $x$ or $y$ from second factor

... 

Pick either $x$ or $y$ from $n$th factor

How many ways to get exactly $k$ $x$’s?
an identity with binomial coefficients

\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n \]

Proof:

\[ \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1 + 1)^n = 2^n \]
How many ways to walk from 1st and Spring to 5th and Pine only going North and East?

A: *Changing the visualization often helps.* Instead of tracing paths on the grid above, list choices. You walk 7 blocks; at each intersection choose N or E; must choose N exactly 3 times.

\[
\binom{7}{3} = 35
\]
How many ways to walk from 1\textsuperscript{st} and Spring to 5\textsuperscript{th} and Pine only going North and East, if I want to stop at Starbucks on the way?
10 people of different heights. How many ways to line up 5 of them?

Line up 5 of them in height order?

# of ways to rearrange letters in word SYSTEMS
# of 7 digit numbers (decimal) with at least one repeating digit? (allowed to have leading zeros).