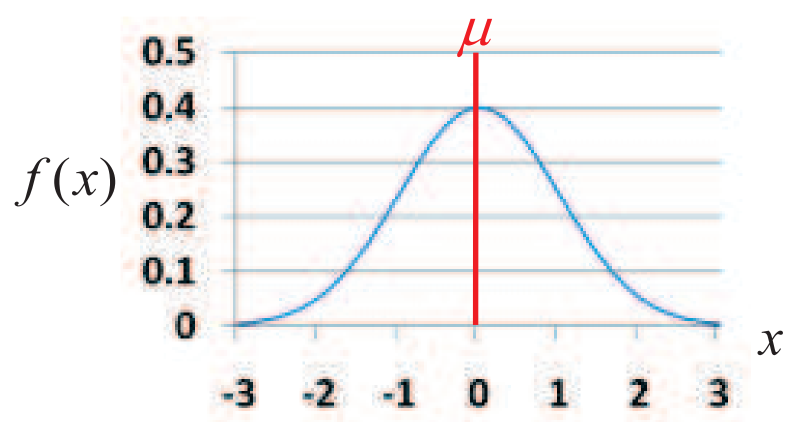
# Normal Random Variable

- $X$ is a **<u>Normal Random Variable</u>**: $X \sim N(\mu, \sigma^2)$
  - Probability Density Function (PDF):

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{where} \quad -\infty < x < \infty$$

  - $E[X] = \mu$
  - $Var(X) = \sigma^2$




  - Also called "Gaussian"
  - Note: $f(x)$ is symmetric about $\mu$
  - Common for natural phenomena: heights, weights, etc.
  - Often results from the sum of multiple variables

# Carl Friedrich Gauss

- Carl Friedrich Gauss (1777-1855) was a remarkably influential German mathematician

- Started doing groundbreaking math as teenager
  - Did not invent Normal distribution, but popularized it
- He looked more like Martin Sheen
  - Who is, of course, Charlie Sheen's father

# Properties of Normal Random Variable

- Let $X \sim N(\mu, \sigma^2)$
- Let $Y = aX + b$
  - $Y \sim N(a\mu + b, a^2\sigma^2)$
  - $E[Y] = E[aX + b] = aE[X] + b = a\mu + b$
  - $Var(Y) = Var(aX + b) = a^2 Var(X) = a^2\sigma^2$

$$F_Y(x) = P(Y \le x) = P(aX + b \le x) = P(X \le \tfrac{x-b}{a}) = F_X(\tfrac{x-b}{a})$$

Differentiating $F_Y(x)$ w.r.t. $x$ , yields $f_Y(x)$, the PDF for $y$:

$$f_Y(x) = \frac{d}{dx} F_Y(x) = \frac{d}{dx} F_X(\tfrac{x-b}{a}) = \frac{1}{a} f_X(\tfrac{x-b}{a})$$

- Special case: $Z = (X - \mu)/\sigma$ $(a = 1/\sigma, b = -\mu/\sigma)$
  - $Z \sim N(a\mu + b, a^2\sigma^2) = N(\mu/\sigma - \mu/\sigma, (1/\sigma)^2\sigma^2) = N(0, 1)$

# Standard (Unit) Normal Random Variable

- $Z$ is a **Standard (or Unit) Normal RV**: $Z \sim N(0, 1)$
  - E[Z] = $\mu$ = 0     Var(Z) = $\sigma^2$ = 1     SD(Z) = $\sigma$ = 1
  - CDF of Z, $F_Z(z)$ does not have closed form
  - We denote $F_Z(z)$ as $\Phi(z)$: "phi of z"

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

  - By symmetry: $\Phi(-z) = P(Z \le -z) = P(Z \ge z) = 1 - \Phi(z)$
- Use Z to compute $X \sim N(\mu, \sigma^2)$, where $\sigma > 0$

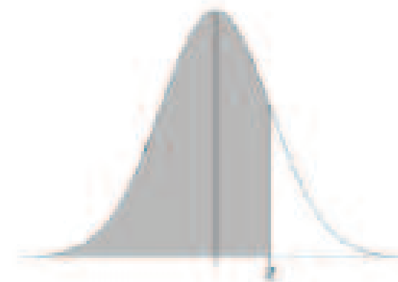
$$F_X(x) = P(X \le x) = P(\tfrac{X-\mu}{\sigma} \le \tfrac{x-\mu}{\sigma}) = P(Z \le \tfrac{x-\mu}{\sigma}) = \Phi(\tfrac{x-\mu}{\sigma})$$

  - Table of $\Phi(z)$ values in textbook, p. 201 and handout

# Using Table of $\Phi(z)$ Values

**Standard Normal Cumulative Probability Table**

$\Phi(0.54) = 0.7054$



Cumulative probabilities for **POSITIVE** z-values are shown in the following table:

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |

# Get Your Gaussian On

- X ~ N(3, 16)   $\mu = 3$   $\sigma^2 = 16$   $\sigma = 4$
  - What is P(X > 0)?

$$P(X > 0) = P(\frac{X-3}{4} > \frac{0-3}{4}) = P(Z > -\frac{3}{4})$$

$$1 - \Phi(-\tfrac{3}{4}) = \Phi(\tfrac{3}{4}) = 0.7734$$

  - What is P(2 < X < 5)?

$$P(2 < X < 5) = P(\frac{2-3}{4} < \frac{X-3}{4} < \frac{5-3}{4}) = P(-\frac{1}{4} < Z < \frac{2}{4})$$

$$\Phi(\tfrac{2}{4}) - \Phi(-\tfrac{1}{4}) = \Phi(\tfrac{1}{2}) - (1 - \Phi(\tfrac{1}{4})) = 0.6915 - (1 - 0.5987) = 0.2902$$

  - What is P(|X – 3| > 6)?

$$P(X < -3) + P(X > 9) = P(Z < \frac{-3-3}{4}) + P(Z > \frac{9-3}{4})$$

$$\Phi(-\tfrac{3}{2}) + (1 - \Phi(\tfrac{3}{2})) = 2(1 - \Phi(\tfrac{3}{2})) = 2(1 - 0.9332) = 0.1336$$

# Noisy Wires

- Send voltage of 2 or -2 on wire (to denote 1 or 0)
  - X = voltage sent
  - R = voltage received = X + Y, where noise Y ~ N(0, 1)
  - Decode R: if (R ≥ 0.5) then 1, else 0
  - What is P(error after decoding | original bit = 1)?

$$P(2+Y<0.5)=P(Y<-1.5)=\Phi(-1.5)=1-\Phi(1.5)\approx 0.0668$$

  - What is P(error after decoding | original bit = 0)?

$$P(-2+Y\geq 0.5)=P(Y\geq 2.5)=1-\Phi(2.5)\approx 0.0062$$

# Normal Approximation to Binomial

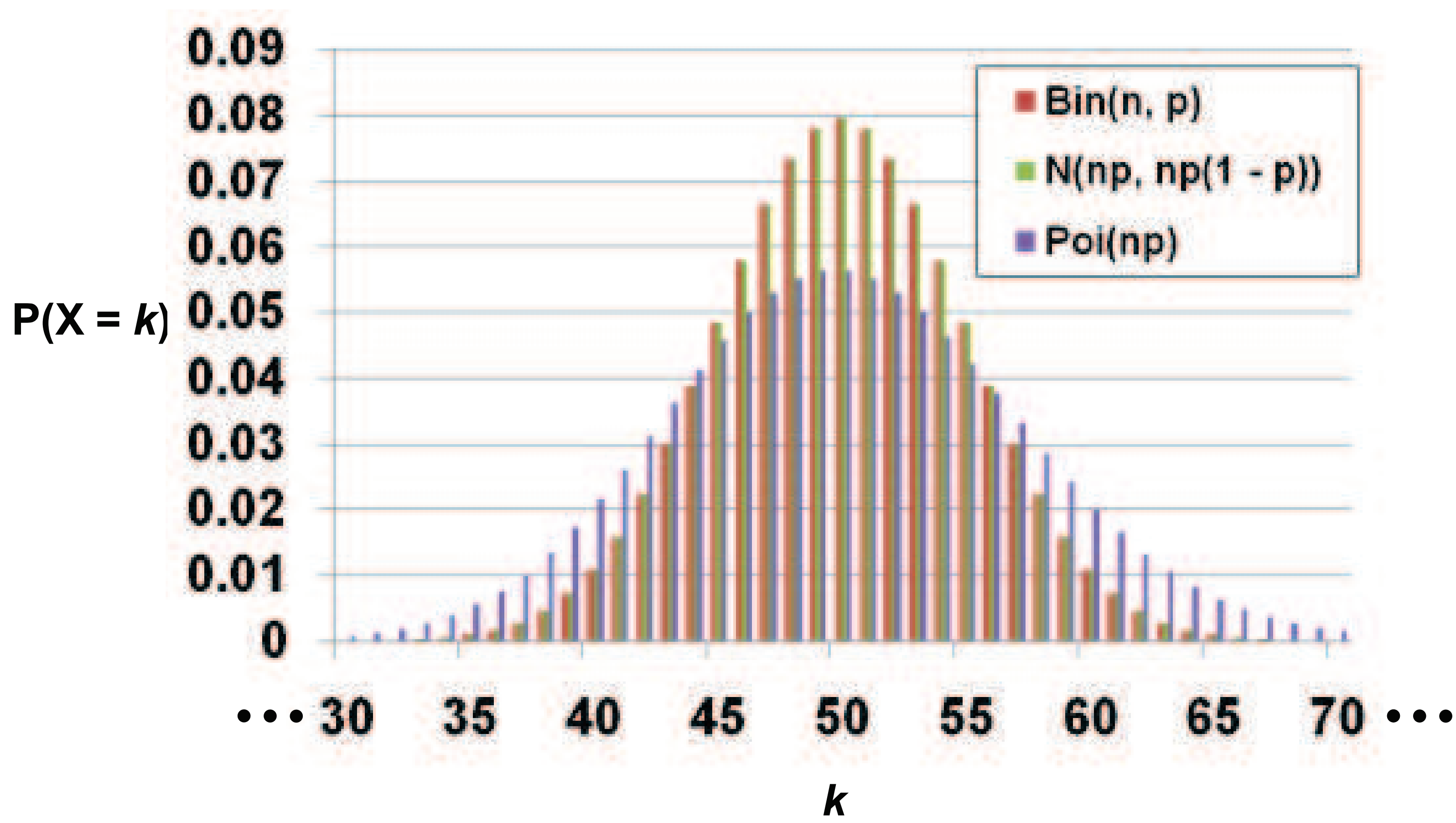
- X ~ Bin($n$, $p$)
  - E[X] = $np$ Var(X) = $np(1 - p)$
  - Poisson approx. good: $n$ large (> 20), $p$ small (< 0.05)
  - For large $n$: X ≈ Y ~ N(E[X], Var(X)) = N($np$, $np(1 - p)$)
  - Normal approx. good : Var(X) = $np(1 - p) \geq 10$

$$P(X = k) \approx P\left(k - \tfrac{1}{2} < Y < k + \tfrac{1}{2}\right) = \Phi\left(\frac{k - np + 0.5}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k - np - 0.5}{\sqrt{np(1-p)}}\right)$$

  - DeMoivre-Laplace Limit Theorem:
    - $S_n$: number of successes (with prob. $p$) in $n$ independent trials

$$P\left(a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\right) \xrightarrow{n \to \infty} \Phi(b) - \Phi(a)$$

# Comparison when $n = 100$, $p = 0.5$



# Faulty Endorsements

- 100 people placed on special diet
  - X = # people on diet whose cholesterol decreases
  - Doctor will endorse diet if X ≥ 65
  - What is P(doctor endorses diet | diet has no effect)?
  - X ~ Bin(100, 0.5)

$$np = 50 \quad np(1-p) = 25 \quad \sqrt{np(1-p)} = 5$$

  - Use Normal approximation: Y ~ N(50, 25)

$$P(X \geq 65) \approx P(Y > 64.5)$$

$$P(Y \geq 64.5) = P\left(\frac{Y-50}{5} > \frac{64.5-50}{5}\right) = 1 - \Phi(2.9) \approx 0.0019$$

  - Using Binomial:

$$P(X \geq 65) \approx 0.0018$$

# Stanford Admissions

- Stanford accepts 2480 students
  - Each accepted student has 68% chance of attending
  - X = # students who will attend.   X ~ Bin(2480, 0.68)
  - What is P(X > 1745)?

$$np = 1686.4 \quad np(1-p) \approx 539.65 \quad \sqrt{np(1-p)} \approx 23.23$$

  - Use Normal approximation: Y ~ N(1686.4, 539.65)

$$P(X > 1745) \approx P(Y \geq 1745.5)$$

$$P(Y \geq 1745.5) = P\left(\frac{Y - 1686.4}{23.23} > \frac{1745.5 - 1686.4}{23.23}\right) = 1 - \Phi(2.54) \approx 0.0055$$

  - Using Binomial:

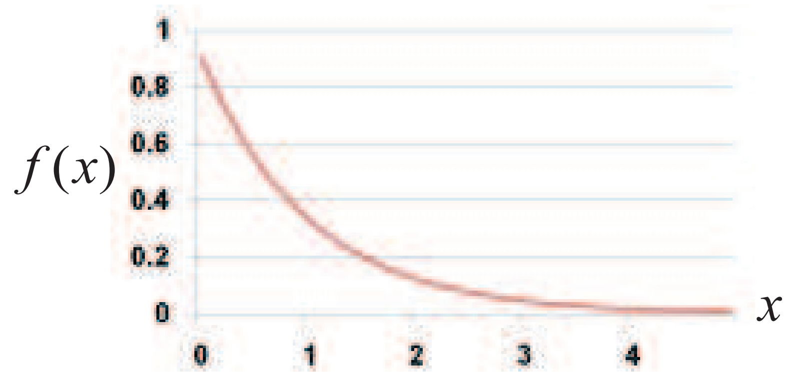
$$P(X > 1745) \approx 0.0053$$

# Exponential Random Variable

- $X$ is an **<u>Exponential RV</u>**: $X \sim \text{Exp}(\lambda)$   Rate: $\lambda > 0$
  - Probability Density Function (PDF):

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{where} \quad -\infty < x < \infty$$

  - $E[X] = \dfrac{1}{\lambda}$
  - $Var(X) = \dfrac{1}{\lambda^2}$




  - Cumulative distribution function (CDF), $F$(X) = P($X \leq x$):

$$F(x) = 1 - e^{-\lambda x} \quad \text{where } x \geq 0$$

  - Represents time until some event
    - Earthquake, request to web server, end cell phone contract, etc.

# Exponential is "Memoryless"

- X = time until some event occurs
  - $X \sim \text{Exp}(\lambda)$
  - What is $P(X > s + t \mid X > s)$?

$$P(X > s+t \mid X > s) = \frac{P(X > s+t \text{ and } X > s)}{P(X > s)} = \frac{P(X > s+t)}{P(X > s)}$$

$$\frac{P(X > s+t)}{P(X > s)} = \frac{1 - F(s+t)}{1 - F(s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = 1 - F(t) = P(X > t)$$

$$\text{So, } P(X > s+t \mid X > s) = P(X > t)$$

  - After initial period of time *s*, $P(X > t \mid \bullet)$ for waiting another *t* units of time until event is same as at start
  - "Memoryless" = no impact from preceding period *s*

# Visits to Web Site

- Say a visitor to your web leaves after X minutes
  - On average, visitors leave site after 5 minutes
  - Assume length of stay is Exponentially distributed
  - X ~ Exp($\lambda$ = 1/5), since E[X] = 1/$\lambda$ = 5
  - What is P(X > 10)?

$$P(X > 10) = 1 - F(10) = 1 - (1 - e^{-\lambda 10}) = e^{-2} \approx 0.1353$$

  - What is P(10 < X < 20)?

$$P(10 < X < 20) = F(20) - F(10) = (1 - e^{-4}) - (1 - e^{-2}) \approx 0.1170$$

# Replacing Your Laptop

- X = # hours of use until your laptop dies
  - On average, laptops die after 5000 hours of use
  - X ~ Exp($\lambda$ = 1/5000), since E[X] = 1/$\lambda$ = 5000
  - You use your laptop 5 hours/day.
  - What is P(your laptop lasts 4 years)?
  - That is: P(X > (5)(365)(4) = 7300)

$$P(X > 7300) = 1 - F(7300) = 1 - (1 - e^{-7300/5000}) = e^{-1.46} \approx 0.2322$$

  - Better plan ahead... especially if you are coterming:

$$P(X > 9125) = 1 - F(9125) = e^{-1.825} \approx 0.1612 \quad \text{(5 year plan)}$$

$$P(X > 10950) = 1 - F(10950) = e^{-2.19} \approx 0.1119 \quad \text{(6 year plan)}$$

# A Little Calculus Review

- Product rule for derivatives:

$$d(u \cdot v) = du \cdot v + u \cdot dv$$

- Derivative and integral of exponential:

$$\frac{d(e^u)}{dx} = e^u \frac{du}{dx} \qquad \int e^u du = e^u$$

- Integration by parts:

$$\int d(u \cdot v) = u \cdot v = \int v \cdot du + \int u \cdot dv$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

# And Now, Some Calculus Practice

- Compute *n*-th moment of Exponential distribution

$$E[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x} dx$$

  - Step 1: don't panic, think happy thoughts, recall...
  - Step 2: find *u* and *v* (and *du* and *dv*):

$$u = x^n \qquad v = -e^{-\lambda x}$$

$$du = nx^{n-1}dx \qquad dv = \lambda e^{-\lambda x}dx$$

  - Step 3: substitute (a.k.a. "plug and chug")

$$\int u \cdot dv = \int x^n \cdot \lambda e^{-\lambda x} dx = u \cdot v - \int v \cdot du = -x^n e^{-\lambda x} + \int nx^{n-1} e^{-\lambda x} dx$$

$$E[X^n] = -x^n e^{-\lambda x}\Big|_0^\infty + \int nx^{n-1} e^{-\lambda x} dx = 0 + \frac{n}{\lambda}\int x^{n-1}\lambda e^{-\lambda x} dx = \frac{n}{\lambda} E[X^{n-1}]$$

$$\text{Base case}: E[X^0] = E[1] = 1, \quad \text{so } E[X] = \frac{1}{\lambda}, E[X^2] = \frac{2}{\lambda}\frac{1}{\lambda} = \frac{2}{\lambda^2}, \ldots$$