Expected value of random variable X given event A

$$E(X|A) = \sum_{x \in Range(X)} x Pr(X = x|A)$$

Law of Total Expectation (example)

49.8% of population male

Average height 5'11" (men) 5'5" (female)

$$E(H) = E(H|M)Pr(M) + E(H|F)Pr(F)$$
$$= 5\frac{11}{12} \cdot 0.498 + 5\frac{5}{12} \cdot 0.502$$

Law of Total Expectation

X random variable on a sample space S

$A_1, A_2, \ldots, A_k$ partition of S

$$\begin{aligned}
E(X) &= \sum_i E(X|A_i)Pr(A_i) \\
&= \sum_i \sum_x xPr(X = x|A_i)Pr(A_i) \\
&= \sum_x \sum_i xPr(X = x|A_i)Pr(A_i) \\
&= \sum_x x \sum_i Pr(X = x|A_i)Pr(A_i) \\
&= \sum_i xPr(X = x)
\end{aligned}$$

Law of Total Expectation : Application

System that fails in step i independently with probability p

X # steps to fail

E(X) ?

Let A be the event that system fails in first step.

$$E(X) = E(X|A)Pr(A) + E(X|\overline{A})Pr(\overline{A})$$

$$= p + (1 + E(X))(1 - p)$$

$$= 1 + (1 - p)E(X)$$

$$E(X) = \frac{1}{p}$$

Law of Total Expectation : Example

A miner is trapped in a mine containing 3 doors.

- The 1st door leads to a tunnel that will take him to safety after 3 hours.
- The 2nd door leads to a tunnel that returns him to the mine after 5 hours.
- The 3rd door leads to a tunnel that returns him to the mine after 7 hours.

At all times, he is equally likely to choose any one of the doors.

E(time to reach safety) ?

Problem

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all the passengers.

Game of Craps

- Begin by rolling an ordinary pair of dice
- If the sum of dice is 2, 3 or 12, the player loses
- If the sum of dice is 7 or 11, the player wins
- If it is any other number, say k, the player continues to roll the dice until the sum is either 7 or k.
    - If it is 7, the player loses.
    - If it is k, the player wins.

Let R denote the number of rolls of the dice in a game of craps.

- What is E(R)?
- What is E(R | player wins)?
- What is E(R | player loses)?