Continuous Random Variables

Seen discrete r.v.'s

(Ω, P) prob space

$X : \Omega \to \mathbb{R}$ e.g. integer-valued $\text{Range}(X) = x_1, \ldots, x_k$

p.m.f. $p_X(x_i) = \Pr(X = x_i)$

$$F_X(x) = \sum_{i \mid x_i \le x} p_X(x_i)$$

C.D.F. $F_X(x) = \Pr(X \le x)$

$$p_X(x_i) = F_X(x_i) - F_X(x_{i-1})$$

Suppose we want to represent continuous valued random var

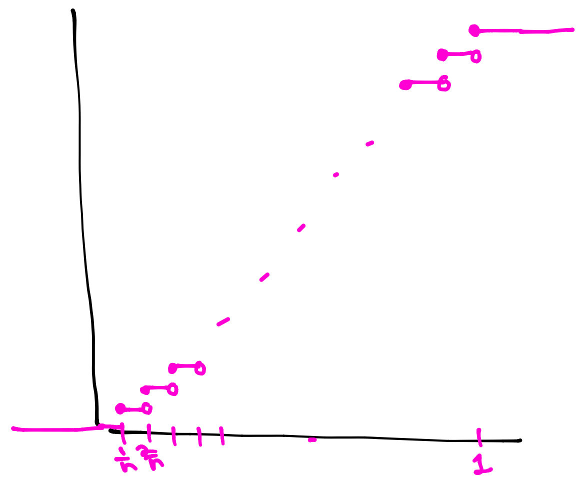
e.g. draw of random $x \in (0,1]$

Discrete approx: $\Omega = \left\{ \frac{1}{n}, \frac{2}{n}, \ldots \frac{n-1}{n}, \frac{n}{n} \right\}$

$$p_X(x) = \begin{cases} \frac{1}{n} & x = \frac{i}{n} \quad i = 1..n \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \frac{j}{n} \qquad \frac{j}{n} \le x < \frac{j+1}{n} \qquad j = 0..n-1$$

$$\lim_{n\to\infty} F_X(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x \le 1 \\ 1 & x > 1 \end{cases}$$



$$\lim_{n\to\infty} p_X(x) = 0$$

notion of prob mass fn doesn't make sense anymore

Instead $\boxed{\text{probability density function}}$

$$f_X(x) = \frac{d}{dx} F_X(x) \qquad F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

Recall discrete setting $F_X(x) = \sum_{y \le x} p_X(y)$

Properties:

- $F_X(x)$ monotone $\uparrow$ from 0 to 1
- $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ $\qquad f_X(x) \ge 0$ (but not necessarily ≤ 1)
- $\Pr(a \le X \le b) = \int_a^b f_X(x)\,dx$

Continuous setting

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

Discrete setting

$$E(X) = \sum_{x \in \text{Range}(X)} x\, p(x)$$

$$E(X) = \sum_{x \in \text{Range}(X)} g(x)\, p(x)$$

Example: X cont. r.v.

$$f(x) = \begin{cases} C(4x-2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

1) What is C?

$$\int_0^2 C(4x-2x^2)\,dx = 1 \qquad 2C\left[x^2 - \frac{x^3}{3}\right]\Big|_0^2 = 8C\left[1-\frac{2}{3}\right] = \frac{8}{3}C$$

$$\Rightarrow C = \frac{3}{8}$$

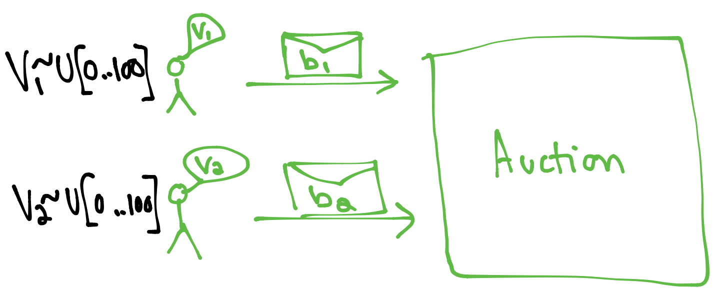
2) What is $Pr(X>1)$?

$$\int_1^2 \frac{3}{8}(4x-2x^2)\,dx$$

3) What is $E(X)$?

$$E(X) = \frac{3}{8}\int_0^2 x\left(4x-2x^2\right)dx$$

Application: Auctions



winner : highest bidder

payment

1^{st} price auction: winner pays his bid

2^{nd} price auction: winner pays 2^{nd} highest bid

How does bidder decide how to bid?

asks: how happy do I expect to be with this bid?

$$\text{happiness} = \begin{cases} \text{value} - \text{price} & \text{if win} \\ 0 & \text{otherwise} \end{cases}$$

bids "best response" to how other guy bids

Ex: $v_1 = \$80$ $\quad b_2 = \$60$

1^{st} price auction analysis

Suppose I promise you the other guy will bid $B_2 = \frac{V_2}{2}$

$V_2 \sim U[0,100]$

How should you bid when your value is v

choose bid b to maximize expected payoff

$$(v-b)\Pr(\text{win}) = (v-b)\Pr\left(b > \frac{V_1}{2}\right)$$

$$= (v-b)\Pr(V_1 < 2b)$$

$$= (v-b)\frac{2b}{100}$$

$$\max_b \frac{(v-b)2b}{100} = \max (v-b)b$$

take deriv $\quad -b + (v-b) = 0 \implies b = \frac{v}{2}$

bidding $\frac{v}{2}$ is an equilibrium ← each bidder is playing a best response to bid of other

$$E(\text{auctioneer revenue}) = E\Big[\underbrace{\max\left(\tfrac{V_1}{2}, \tfrac{V_2}{2}\right)}_{X}\Big]$$

$$F_X(x) = \Pr\left[\max\left(\tfrac{V_1}{2}, \tfrac{V_2}{2}\right) \le x\right]$$

$$= \Pr(V_1 \le 2x, V_2 \le 2x)$$

$$= \frac{2x}{100} \cdot \frac{2x}{100}$$

$$= \left(\frac{2}{100}\right)^2 x^2$$

$$\implies f_X(x) = \frac{8}{100^2} x$$

$$= \int_0^{50} x \cdot f(x)\,dx$$

$$= \frac{8}{(100)^2} \int_0^{50} x^2\,dx = \frac{8}{(100)^2} \frac{x^3}{3}\Big|_0^{50}$$

$$= \frac{100}{3}$$

2^{nd} price auction: "dominant strategy" to report truthfully

$$E[\text{auctioneer revenue}] = E[\underbrace{\min(V_1, V_2)}_{Y}]$$

$$1 - F_Y(y) = Pr(\min(V_1, V_2) \geq y)$$

$$= Pr(V_1 \geq y, V_2 \geq y)$$

$$= \left(1 - \frac{x}{100}\right)\left(1 - \frac{x}{100}\right)$$

$$\Rightarrow F_Y(y) = 1 - \left(1 - \frac{x}{100}\right)^2$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{2}{100}\left(1 - \frac{y}{100}\right)$$

$$E[\min(V_1, V_2)] = \int_0^{100} y \, \frac{2}{100}\left(1 - \frac{y}{100}\right) dy$$

$$= \frac{2}{100}\left[\frac{y^2}{2} - \frac{y^3}{300}\right]_0^{100}$$

$$= \frac{100}{3}$$ Same as 1^{st} price!!