X ~ Exp(λ)

The Exponential Density Function

λ = 2

λ = 1

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

f(x)

x

X ~ Exp(λ)
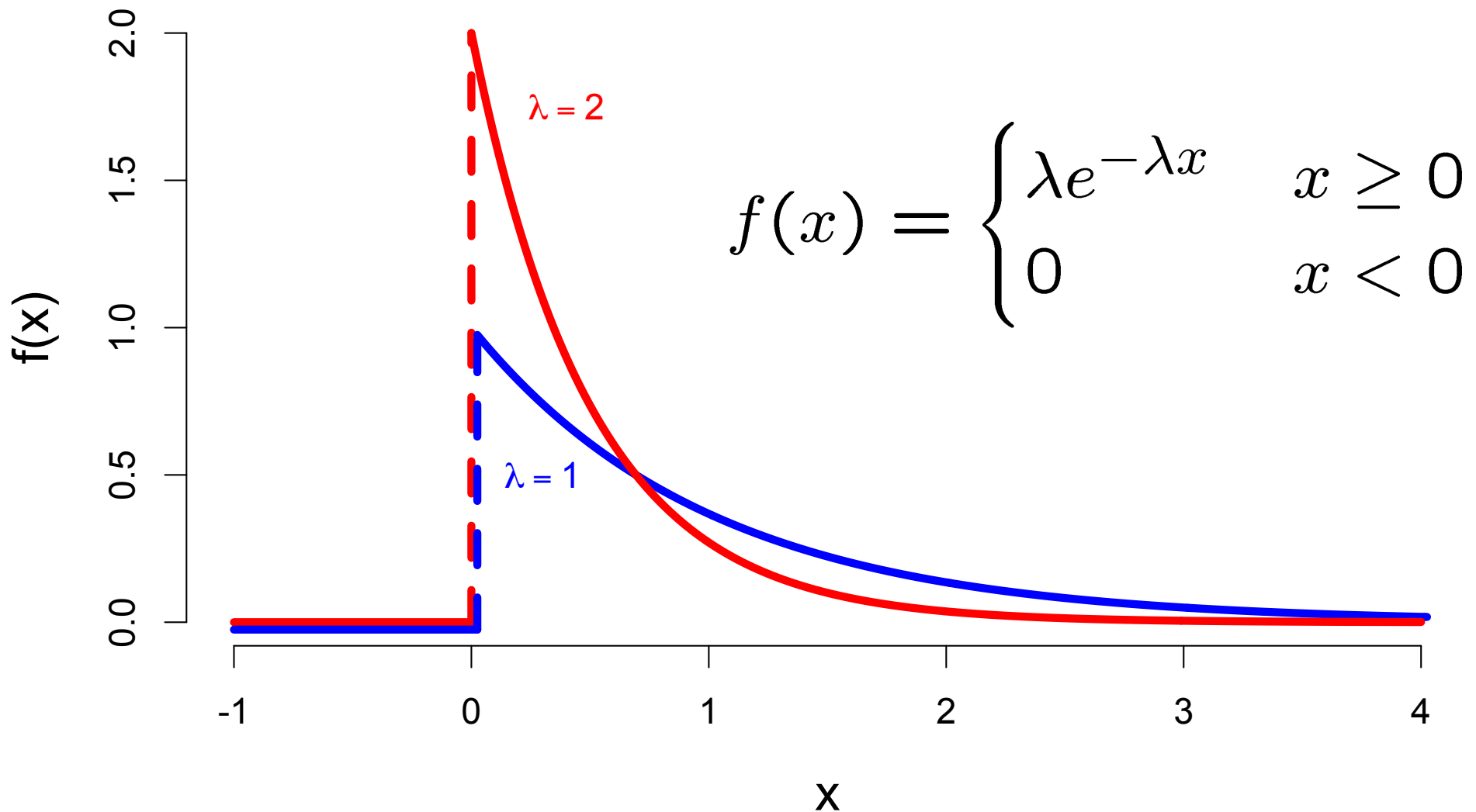
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$E[X] = \frac{1}{\lambda} \qquad \mathrm{Var}[X] = \frac{1}{\lambda^2}$$

$$\Pr(X \geq t) = e^{-\lambda t} = 1\text{-}F(t)$$

Memorylessness:

$$\Pr(X > s + t \mid X > s) = \Pr(X > t)$$

Radioactive decay: How long until the next alpha particle?

Customers: how long until the next customer/packet arrives at the checkout stand/server?

Buses: How long until the next #71 bus arrives on the Ave?

  Yes, they have a schedule, but given the vagaries of traffic, riders with-bikes-and-baby-carriages, etc., can they stick to it?

Relation to the Poisson:

Poisson: *how many* events in a *fixed time*;
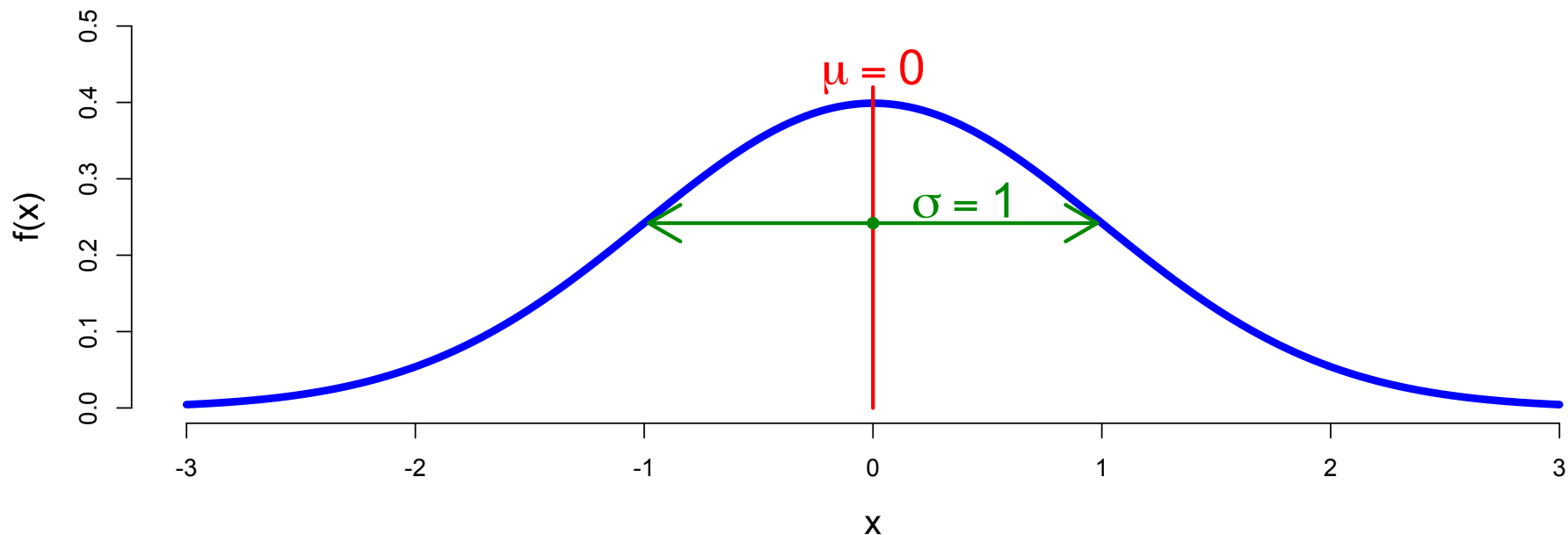Exponential: *how long* until the *next event*

Relation to geometric: Geometric is discrete analog:

X is a normal (aka Gaussian) random variable $X \sim N(\mu, \sigma^2)$

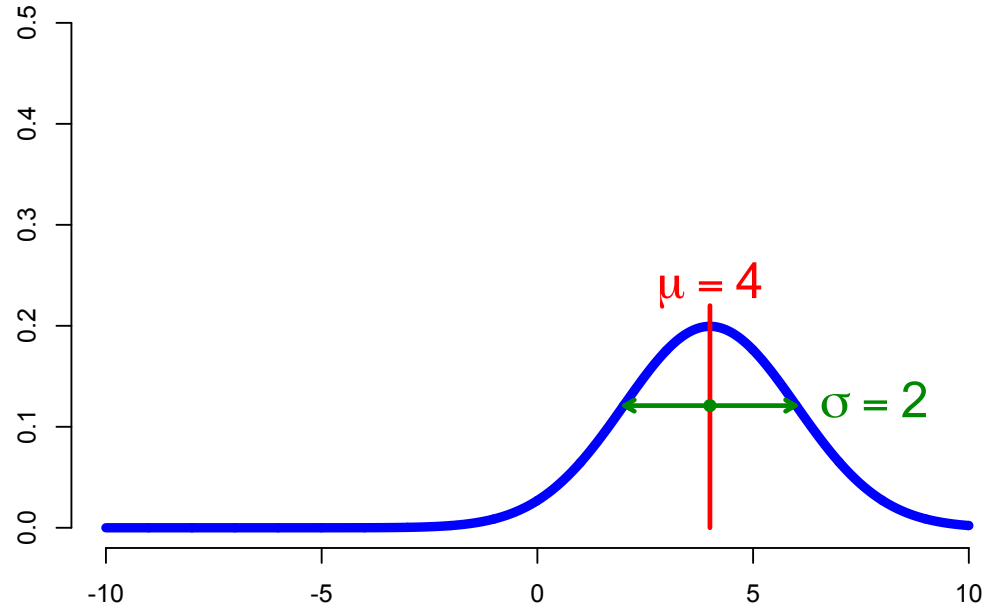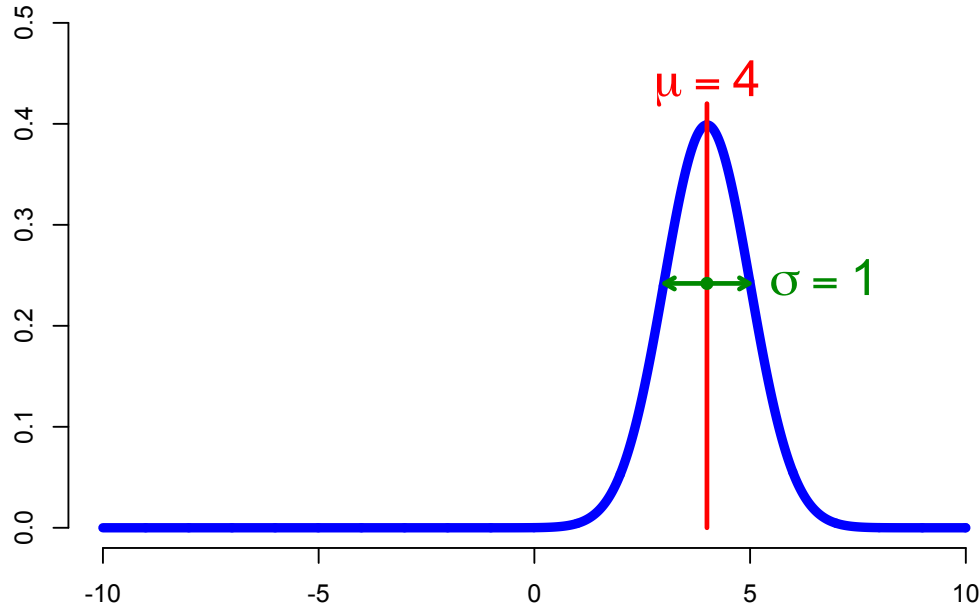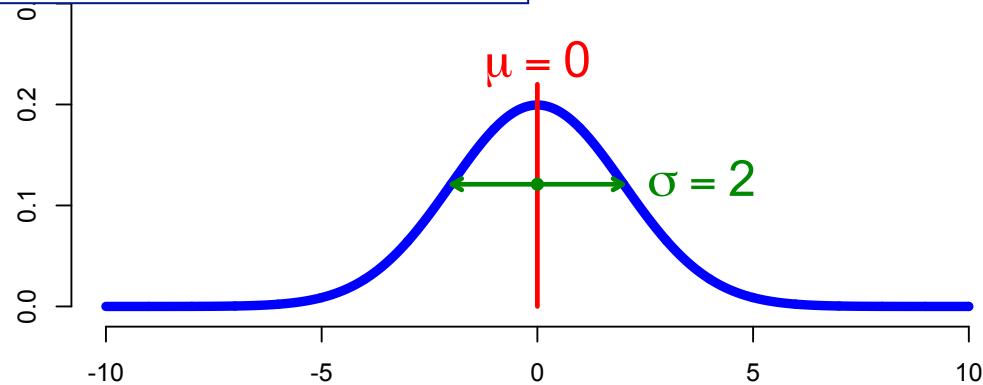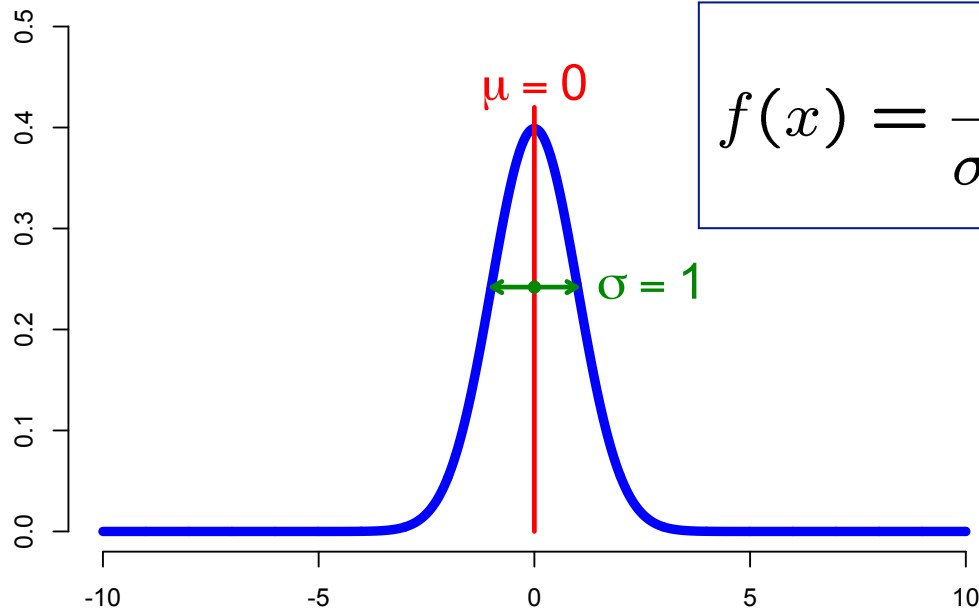$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$E[X] = \mu \qquad \mathrm{Var}[X] = \sigma^2$$

The Standard Normal Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$$

density at μ is $\approx .399/\sigma$

X is a normal random variable $X \sim N(\mu, \sigma^2)$

Y = aX + b
   E[Y] = E[aX+b] = aμ + b
   Var[Y] = Var[aX+b] = $a^2\sigma^2$
   Y $\sim$ N(aμ + b, $a^2\sigma^2$)

Important special case: Z = (X-μ)/σ $\sim$ N(0,1) $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$

Z $\sim$ N(0,1)   "*standard* (or *unit*) normal"
   Use Φ(z) to denote CDF, i.e.

$$\Phi(z) = \Pr(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

   no closed form ☹

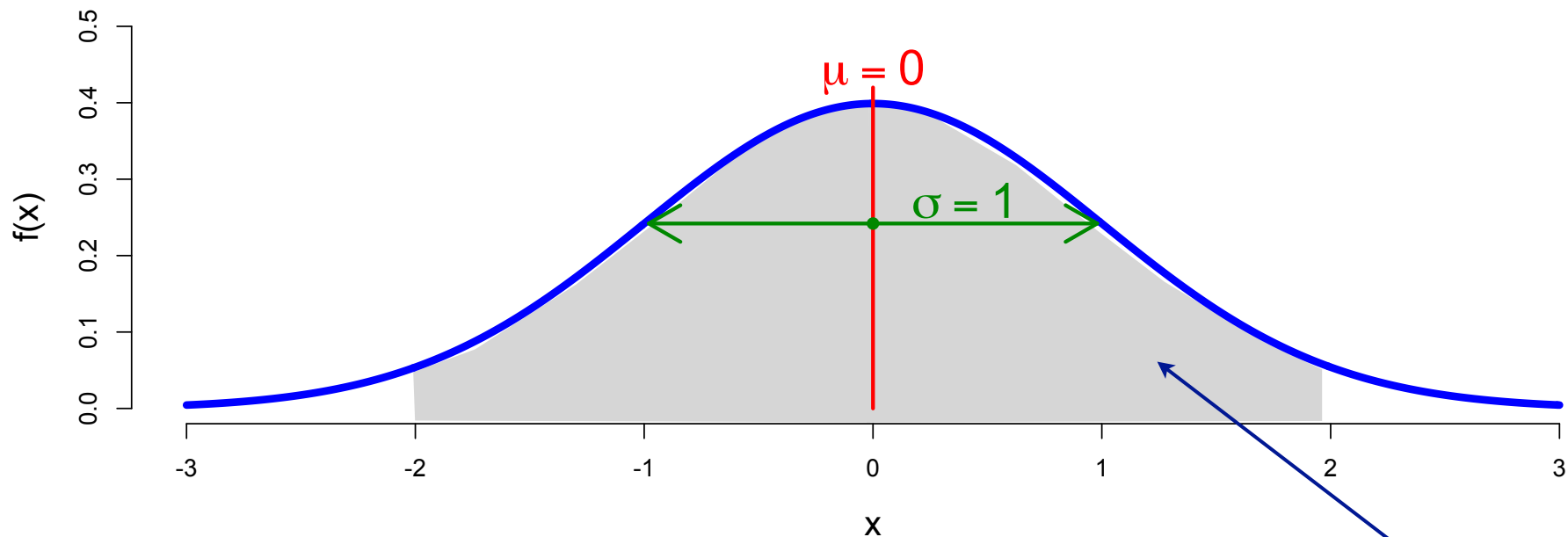# Table of the Standard Normal Cumulative Distribution Function $\Phi(Z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454		0.7517
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764		0.7823
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365
1.0	0.8413	0.8438	0.8461	0.8485					0.8599
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992			0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994			0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997

$\Phi(.46)$

The Standard Normal Density Function

$\mu = 0$

$\sigma = 1$

f(x)
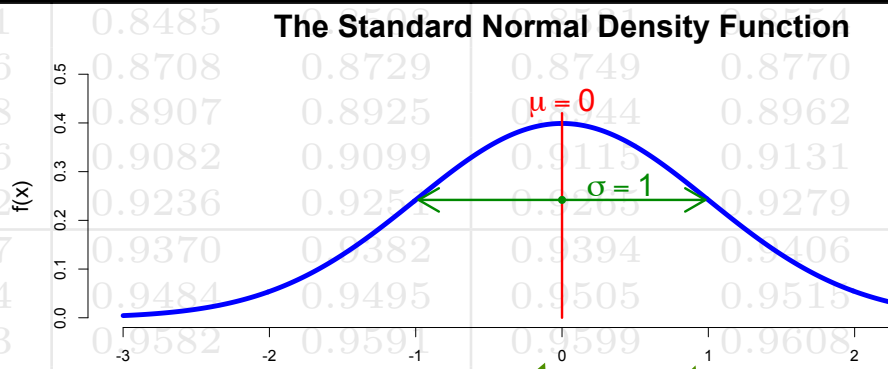
E.g., see B&T p155, p531

The Standard Normal Density Function

If Z ~N(μ,σ) what is P(μ-σ < Z < μ+σ)?

P(μ - σ < Z < μ +   σ) = Φ(1) - Φ(-1) ≈ 68%

P(μ - 2σ < Z < μ + 2σ) = Φ(2) - Φ(-2) ≈ 95%

P(μ - 3σ < Z < μ + 3σ) = Φ(3) - Φ(-3) ≈ 99%

$X \sim \text{Bin}(n,p)$

$E[X] = np \quad \text{Var}[X] = np(1-p)$

Poisson approx: good for n large, p small (np constant)

Normal approx: For large n, (p stays fixed):

$X \approx Y \sim N(E[X], \text{Var}[X]) = N(np, np(1-p))$
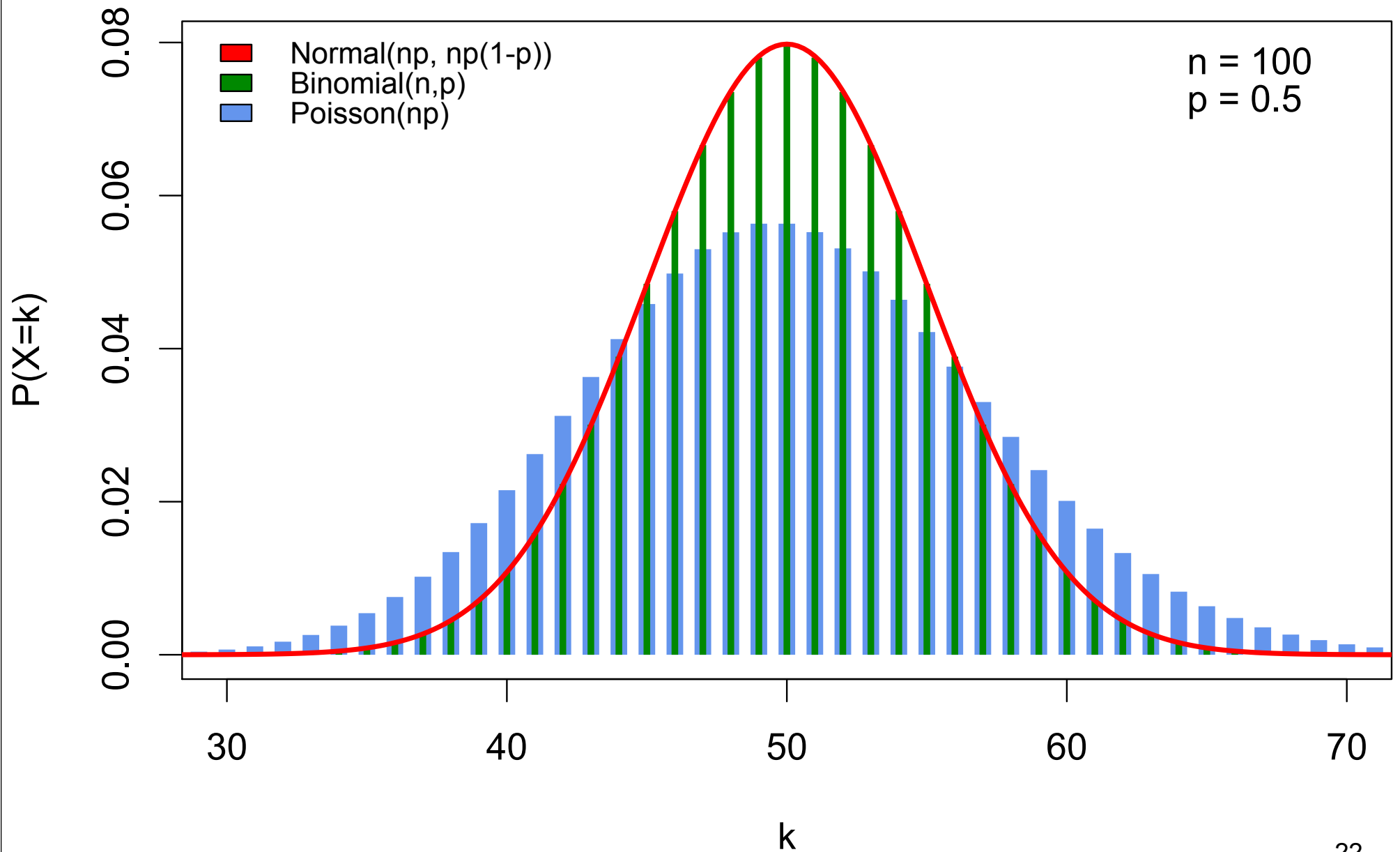
Normal approximation good when $np(1-p) \geq 10$

DeMoivre-Laplace Theorem:

Let S_n = number of successes in n trials (with prob. p).

Then, as $n \rightarrow \infty$:

$$Pr\left(a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\right) \longrightarrow \Phi(b) - \Phi(a)$$

Fair coin flipped 40 times. Probability of 20 heads?

Exact answer:

$$P(X = 20) \quad = \quad \binom{40}{20}\left(\frac{1}{2}\right)^{40} \approx \boxed{0.1254}$$

Normal approximation:

$$P(X = 20) \quad = \quad P(19.5 \leq X < 20.5)$$

$$= \quad P\left(\frac{19.5 - 20}{\sqrt{10}} \leq \frac{X - 20}{\sqrt{10}} < \frac{20.5 - 20}{\sqrt{10}}\right)$$

$$\approx \quad P\left(-0.16 \leq \frac{X - 20}{\sqrt{10}} < 0.16\right)$$

$$\approx \quad \Phi(0.16) - \Phi(-0.16) \approx \boxed{0.1272}$$

pdf and cdf

$$f(x) = \frac{d}{dx} F(x) \qquad F(a) = \int_{-\infty}^{a} f(x) \, dx$$

sums become integrals, e.g.

$$E[X] = \Sigma_x \, xp(x) \qquad E[X] = \int_{-\infty}^{\infty} x \cdot f(x) \, dx$$

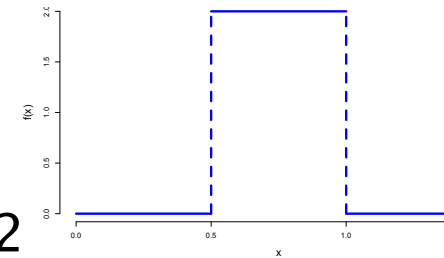most familiar properties still hold, e.g.

$$E[aX+bY+c] = aE[X]+bE[Y]+c$$

$$Var[X] = E[X^2] - (E[X])^2$$

Three important examples

X ~ Uni(α,β) uniform in [α,β]

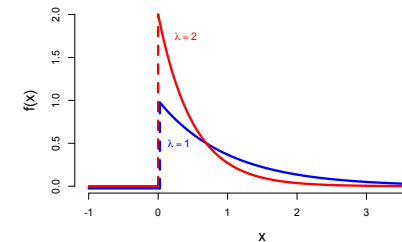$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & x \in [\alpha, \beta] \\ 0 & \text{otherwise} \end{cases}$$

E[X] = (α+β)/2
Var[X] = (α−β)²/12

X ~ Exp(λ) exponential

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$E[X] = \frac{1}{\lambda}$$
$$\text{Var}[X] = \frac{1}{\lambda^2}$$

X ~ N(μ, σ²) normal (aka Gaussian)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$E[X] = \mu$$
$$\text{Var}[X] = \sigma^2$$

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