Discrete probability

Readings: BT 1.1-1.2, Rosen 6.1-6.2
Sample space: \( S \) is the set of all possible outcomes of an experiment (\( \Omega \) in your textbook—Greek uppercase omega)

- Coin flip: \( S = \{ \text{Heads, Tails} \} \)
- Flipping two coins: \( S = \{(H,H), (H,T), (T,H), (T,T)\} \)
- Roll of one 6-sided die: \( S = \{1, 2, 3, 4, 5, 6\} \)
- # emails in a day: \( S = \{ x : x \in \mathbb{Z}, x \geq 0 \} \)
- YouTube hrs. in a day: \( S = \{ x : x \in \mathbb{R}, 0 \leq x \leq 24 \} \)
**Events:** \( E \subseteq S \) is some subset of the sample space

- Coin flip is heads: \( E = \{\text{Head}\} \)
- At least one head in 2 flips: \( E = \{(H,H), (H,T), (T,H)\} \)
- Roll of die is 3 or less: \( E = \{1, 2, 3\} \)
- # emails in a day < 20: \( E = \{x : x \in \mathbb{Z}, \ 0 \leq x < 20\} \)
- Wasted day (>5 YT hrs): \( E = \{x : x \in \mathbb{R}, \ x > 5 \} \)
set operations on events

E and F are events in the sample space $S$
set operations on events

E and F are events in the sample space $S$

Event “E OR F”, written $E \cup F$

$S = \{1,2,3,4,5,6\}$
outcome of one die roll

$E = \{1,2\}$, $F = \{2,3\}$

$E \cup F = \{1,2,3\}$
set operations on events

E and F are events in the sample space S

Event “E AND F”, written $E \cap F$ or EF

$S = \{1,2,3,4,5,6\}$
outcome of one die roll

$E = \{1,2\}$, $F = \{2,3\}$

$E \cap F = \{2\}$
set operations on events

E and F are events in the sample space $S$

$E F = \emptyset \iff$ E,F are “mutually exclusive”

$S = \{1,2,3,4,5,6\}$

outcome of one die roll

$E = \{1,2\}$, $F = \{2,3\}$, $G=\{5,6\}$

$E F = \{2\}$, not mutually exclusive, but E,G and F,G are
E and F are events in the sample space $S$.

Event “not E,” written $\bar{E}$ or $\neg E$.

- $S = \{1, 2, 3, 4, 5, 6\}$, outcome of one die roll.
- $E = \{1, 2\}$
- $\neg E = \{3, 4, 5, 6\}$
DeMorgan's Laws

\[ \overline{E \cup F} = \overline{E} \cap \overline{F} \]

\[ \overline{E \cap F} = \overline{E} \cup \overline{F} \]
axioms of probability

Intuition: Probability as the relative frequency of an event

\[ \text{Pr}(E) = \lim_{n \to \infty} \left( \frac{\text{# of occurrences of } E \text{ in } n \text{ trials}}{n} \right) \]

Axiom 1: \( 0 \leq \text{Pr}(E) \leq 1 \)

Axiom 2: \( \text{Pr}(S) = 1 \)

Axiom 3: If \( E \) and \( F \) are mutually exclusive (\( EF = \emptyset \)), then

\[ \text{Pr}(E \cup F) = \text{Pr}(E) + \text{Pr}(F) \]

For any sequence \( E_1, E_2, \ldots, E_n \) of mutually exclusive events,

\[ \text{Pr} \left( \bigcup_{i=1}^{n} E_i \right) = \text{Pr}(E_1) + \cdots + \text{Pr}(E_n) \]
- $\Pr(\overline{E}) = 1 - \Pr(E)$
  \[
  \Pr(\overline{E}) = \Pr(S) - \Pr(E) \text{ because } S = E \cup \overline{E}
  \]

- If $E \subseteq F$, then $\Pr(E) \leq \Pr(F)$
  \[
  \Pr(F) = \Pr(E) + \Pr(F - E) \geq \Pr(E)
  \]

- $\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(EF)$
  
  inclusion-exclusion formula

- And many others
Simplest case: sample spaces with equally likely outcomes.

- **Coin flips:** \( S = \{\text{Heads, Tails}\} \)
- **Flipping two coins:** \( S = \{(H,H),(H,T),(T,H),(T,T)\} \)
- **Roll of 6-sided die:** \( S = \{1, 2, 3, 4, 5, 6\} \)

\[
\text{Pr(each outcome)} = \frac{1}{|S|}
\]

In that case,

\[
\text{Pr}(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|}
\]
Roll two 6-sided dice. What is \( \Pr(\text{sum of dice} = 7) \) ?

\[
S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}
\]

\[
E = \{ (6,1), (5,2), (4,3), (3,4), (2,5), (1,6) \}
\]

\[
\Pr(\text{sum} = 7) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}.
\]
twinkies and ding dongs
4 Twinkies and 3 DingDongs in a bag. 3 drawn. All outcomes equally likely. What is \( \Pr(\text{one Twinkie and two DingDongs drawn}) \)?

**Ordered:**
- Pick 3 ordered options: \(|S| = 7 \cdot 6 \cdot 5 = 210\)
- Pick Twinkie as either 1\(^{st}\), 2\(^{nd}\), or 3\(^{rd}\) item:
  \[ |E| = (4 \cdot 3 \cdot 2) + (3 \cdot 4 \cdot 2) + (3 \cdot 2 \cdot 4) = 72 \]
- \( \Pr(\text{1 Twinkie and 2 DingDongs}) = \frac{72}{210} = \frac{12}{35} \).

**Unordered:**
- \(|S| = \binom{7}{3} = 35\)
- \(|E| = \binom{4}{1} \binom{3}{2} = 12\)
- \( \Pr(\text{1 Twinkie and 2 DingDongs}) = \frac{12}{35} \).
birthdays
What is the probability that, of \( n \) people, none share the same birthday?

\[
|S| = (365)^n \\
|E| = (365)(364)(363) \cdots (365-n+1) \\
Pr(\text{no matching birthdays}) = \frac{|E|}{|S|} = \frac{(365)(364) \cdots (365-n+1)}{(365)^n}
\]

Some values of \( n \)...

- \( n = 23 \): \( Pr(\text{no matching birthdays}) < 0.5 \)
- \( n = 77 \): \( Pr(\text{no matching birthdays}) < 1/5000 \)
- \( n = 100 \): \( Pr(\text{no matching birthdays}) < 1/3,000,000 \)
- \( n = 150 \): \( Pr(\ldots) < 1/3,000,000,000,000,000,000 \)
n = 366?

Pr = 0

Above formula gives this, since

\[
\frac{(365)(364)\ldots(365-n+1)}{(365)^n} = 0
\]

when n = 366 (or greater).

Even easier to see via pigeon hole principle.
What is the probability that, of n people, none share the same birthday as you?

$|S| = (365)^n$

$|E| = (364)^n$

$Pr(\text{no birthdays matches yours}) = \frac{|E|}{|S|}$

$= (364)^n / (365)^n$

Some values of n…

$n = 23$: $Pr(\text{no matching birthdays}) \approx 0.9388$

$n = 77$: $Pr(\text{no matching birthdays}) \approx 0.8096$

$n = 253$: $Pr(\text{no matching birthdays}) \approx 0.4995$
chip defect detection
n chips manufactured, one of which is defective
k chips randomly selected from n for testing

What is $\text{Pr(defective chip is in } k \text{ selected chips)}$?

$|S| = \binom{n}{k} \quad |E| = \binom{1}{1} \binom{n-1}{k-1}$

Pr(defective chip is in k selected chips)

$$= \frac{\binom{1}{1} \binom{n-1}{k-1}}{\binom{n}{k}} = \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{n!}{k!(n-k)!} = \frac{k}{n}$$
n chips manufactured, one of which is defective
k chips randomly selected from n for testing

What is \( \Pr(\text{defective chip is in k selected chips}) \)?

Different analysis:

• Select k chips at random by permuting all n chips and then choosing the first k.
• Let \( E_i = \text{event that } i^{\text{th}} \text{ chip is defective} \).
• Events \( E_1, E_2, \ldots, E_k \) are mutually exclusive
• \( \Pr(E_i) = 1/n \) for \( i=1,2,\ldots,k \)
• Thus \( \Pr(\text{defective chip is selected}) = \Pr(E_1) + \cdots + \Pr(E_k) = k/n. \)
n chips manufactured, \textit{two} of which are defective k chips randomly selected from n for testing

What is Pr(a defective chip is in k selected chips)?

|S| = \binom{n}{k} \quad |E| = (1 \text{ chip defective}) + (2 \text{ chips defective})

= \binom{2}{1} \binom{n-2}{k-1} + \binom{2}{2} \binom{n-2}{k-2}

Pr(a defective chip is in k selected chips)

= \binom{2}{1} \binom{n-2}{k-1} + \frac{\binom{2}{2} \binom{n-2}{k-2}}{\binom{n}{k}}
n chips manufactured, two of which are defective
k chips randomly selected from n for testing

**What is** \( \Pr(\text{a defective chip is in k selected chips}) \)?

**Another approach:**
\[
\Pr(\text{a defective chip is in k selected chips}) = 1 - \Pr(\text{none})
\]
\[
\Pr(\text{none}): \\
|S| = \binom{n}{k}, |E| = \binom{n-2}{k}, \Pr(\text{none}) = \frac{n-2}{\binom{n}{k}}
\]

\[
\Pr(\text{a defective chip is in k selected chips}) = 1 - \frac{n-2}{\binom{n}{k}}
\]

(Same as above? Check it!)
poker hands
Consider 5 card poker hands.

A “straight” is 5 consecutive rank cards of any suit

What is \( \text{Pr}(\text{straight}) \) ?

\[
|S| = \binom{52}{5}
\]

\[
|E| = 10 \cdot \binom{4}{1}^5
\]

\[
\text{Pr}(\text{straight}) = \frac{10 \binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394
\]