Conditional Expectation

Expected value of random variable $X$ given event $A$

$$E(X|A) = \sum_{x \in \text{Range}(X)} xPr(X = x|A)$$

Law of Total Expectation (example)

49.8% of population male

Average height 5’11” (men) 5’5” (female)

$$E(H) = E(H|M)Pr(M) + E(H|F')Pr(F')$$

$$= \frac{11}{12} \cdot 0.498 + \frac{5}{12} \cdot 0.502$$
Law of Total Expectation

X random variable on a sample space S

\[ A_1, A_2, \ldots, A_k \]  partition of S

\[
E(X) = \sum_{i=1}^{k} E(X|A_i)Pr(A_i)
\]

\[
= \sum_{i=1}^{k} \sum_{x} xPr(X = x|A_i)Pr(A_i)
\]

\[
= \sum_{x} \sum_{i=1}^{k} xPr(X = x|A_i)Pr(A_i)
\]

\[
= \sum_{x} x \sum_{i=1}^{k} Pr(X = x|A_i)Pr(A_i)
\]

\[
= \sum_{i=1}^{k} xPr(X = x)
\]
Law of Total Expectation: Application

System that fails in step i independently with probability \( p \)

\( X \)  # steps to fail

\( E(X) \) ?

Let A be the event that system fails in first step.

\[
E(X) = E(X|A)Pr(A) + E(X|A^c)Pr(A^c)
\]

\[
= p + (1 + E(X))(1 - p)
\]

\[
= 1 + (1 - p)E(X)
\]

\[ E(X) = \frac{1}{p} \]
Law of Total Expectation : Example

A miner is trapped in a mine containing 3 doors.
• The 1\textsuperscript{st} door leads to a tunnel that will take him to safety after 3 hours.
• The 2\textsuperscript{nd} door leads to a tunnel that returns him to the mine after 5 hours.
• The 3\textsuperscript{rd} door leads to a tunnel that returns him to the mine after 7 hours.

At all times, he is equally likely to choose any one of the doors.

\textbf{E(time to reach safety) ?}
Algorithms and randomized algorithms

• Binary search: Given a sorted array of $n$ numbers, determine if the array contains the number 153.
• Given an array of unsorted numbers, sort them.
• Given an array of 0’s and 1’s, either $\frac{1}{2}$ of each, or all 1’s. Determine which.

Worst case running time: measure of work algorithm does
Given array of length $n$

If $n = 0$ or $1$, halt

Otherwise, pick element $p$ of array as “pivot”

Split array into subarrays: $< p$, $= p$, $> p$

Recursively sort subarray $< p$

Recursively sort subarray $> p$

Worst case number of comparisons?

What if we use a random pivot?

That makes it a randomized algorithm!