6. random variables

\[ \text{let } X = \text{index of } \]
A random variable $X$ assigns a real number to each outcome in a probability space.

Ex.

Let $H$ be the number of Heads when 20 coins are tossed
Let $T$ be the total of 2 dice rolls
Let $X$ be the number of coin tosses needed to see 1$^{\text{st}}$ head

Note; even if the underlying experiment has “equally likely outcomes,” the associated random variable may not

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$H$</th>
<th>$P(H)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>TT</td>
<td>0</td>
<td>$P(H=0) = 1/4$</td>
</tr>
<tr>
<td>TH</td>
<td>1</td>
<td>$P(H=1) = 1/2$</td>
</tr>
<tr>
<td>HT</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>HH</td>
<td>2</td>
<td>$P(H=2) = 1/4$</td>
</tr>
</tbody>
</table>
20 balls numbered 1, 2, ..., 20
Draw 3 without replacement
Let \( X \) = the maximum of the numbers on those 3 balls
What is \( P(X \geq 17) \)

\[
P(X = 20) = \frac{19}{20} \approx 0.150
\]

\[
P(X = 19) = \frac{18}{20} \approx 0.134
\]

\[\vdots\]

\[
\sum_{i=17}^{20} P(X = i) \approx 0.508
\]

Alternatively:

\[
P(X \geq 17) = 1 - P(X < 17) = 1 - \frac{16}{20} = 0.508
\]
Flip a (biased) coin repeatedly until 1st head observed.
How many flips? Let $X$ be that number.

$P(X=1) = P(H) = p$

$P(X=2) = P(TH) = (1-p)p$

$P(X=3) = P(TTH) = (1-p)^2p$

... 

Check that it is a valid probability distribution:

$$P \left( \bigcup_{i \geq 1} \{X = i\} \right) = \sum_{i \geq 1} (1-p)^{i-1}p = p \sum_{i \geq 0} (1-p)^i = p \frac{1}{1 - (1-p)} = 1$$
A *discrete* random variable is one taking on a countable number of possible values.

Ex:

\[ X = \text{sum of 3 dice}, \quad 3 \leq X \leq 18, X \in \mathbb{N} \]

\[ Y = \text{index of 1st head in seq of coin flips}, \quad 1 \leq Y, Y \in \mathbb{N} \]

\[ Z = \text{largest prime factor of (1+Y)}, \quad Z \in \{2, 3, 5, 7, 11, \ldots\} \]

If \( X \) is a discrete random variable taking on values from a countable set \( T \subseteq \mathbb{R} \), then

\[
p(a) = \begin{cases} 
P(X = a) & \text{for } a \in T \\ 
0 & \text{otherwise} \end{cases}
\]

is called the *probability mass function*. Note: \( \sum_{a \in T} p(a) = 1 \)
Let $X$ be the number of heads observed in $n$ coin flips

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \text{ where } p = P(H)$$

Probability mass function:
The cumulative distribution function for a random variable $X$ is the function $F: \mathbb{R} \to [0, 1]$ defined by

$$F(a) = P[X \leq a]$$

Ex: if $X$ has probability mass function given by:

$$p(1) = \frac{1}{4} \quad p(2) = \frac{1}{2} \quad p(3) = \frac{1}{8} \quad p(4) = \frac{1}{8}$$

$$F(a) = \begin{cases} 
0 & a < 1 \\
\frac{1}{4} & 1 \leq a < 2 \\
\frac{3}{4} & 2 \leq a < 3 \\
\frac{7}{8} & 3 \leq a < 4 \\
1 & 4 \leq a 
\end{cases}$$

NB: for discrete random variables, be careful about “$\leq$” vs “$<$”
For a discrete r.v. $X$ with p.m.f. $p(\cdot)$, the *expectation of $X*$, aka *expected value* or *mean*, is

$$E[X] = \sum_x x p(x)$$

average of random values, weighted by their respective probabilities

For the equally-likely outcomes case, this is just the average of the possible random values of $X$.

For *unequally-likely* outcomes, it is again the average of the possible random values of $X$, *weighted by their respective probabilities*.

**Ex 1:** Let $X =$ value seen rolling a fair die  $p(1), p(2), \ldots, p(6) = 1/6$

$$E[X] = \sum_{i=1}^{6} ip(i) = \frac{1}{6} (1 + 2 + \cdots + 6) = \frac{21}{6} = 3.5$$

**Ex 2:** Coin flip; $X =$ +1 if H (win $1), -1$ if T (lose $1)$

$$E[X] = (1)\cdot p(+1) + (-1)\cdot p(-1) = 1\cdot(1/2) +(-1)\cdot(1/2) = 0$$
For a discrete r.v. $X$ with p.m.f. $p(\bullet)$, the **expectation of $X$**, aka **expected value** or **mean**, is

$$E[X] = \sum_x x p(x)$$

**Another view:** A gambling game. If $X$ is how much you win playing the game once, how much would you expect to win, on average, per game when repeatedly playing?

**Ex 1:** Let $X =$ value seen rolling a fair die $p(1), p(2), ..., p(6) = 1/6$

If you win $X$ dollars for that roll, how much do you expect to win?

$$E[X] = \sum_{i=1}^{6} i p(i) = \frac{1}{6} (1 + 2 + \cdots + 6) = \frac{21}{6} = 3.5$$

**Ex 2:** Coin flip; $X = +1$ if H (win $1), -1$ if T (lose $1)$

$$E[X] = (+1) \cdot p(+1) + (-1) \cdot p(-1) = 1 \cdot (1/2) + (-1) \cdot (1/2) = 0$$

“a fair game”: in repeated play you expect to win as much as you lose. Long term net gain/loss = 0.
Let $X$ be the number of flips up to & including 1\textsuperscript{st} head observed in repeated flips of a biased coin. If I pay you $1 per flip, how much money would you expect to make?

$$P(H) = p; \quad P(T) = 1 - p = q$$

$$p(i) = pq^{i-1}$$

$$E(x) = \sum_{i \geq 1} ip(i) = \sum_{i \geq 1} ipq^{i-1} = p \sum_{i \geq 1} iq^{i-1} \quad (*)$$

A calculus trick:

$$\sum_{i \geq 1} iy^{i-1} = \sum_{i \geq 1} \frac{d}{dy} y^i = \sum_{i \geq 0} \frac{d}{dy} y^i = \frac{d}{dy} \sum_{i \geq 0} y^i = \frac{d}{dy} \frac{1}{1 - y} = \frac{1}{(1 - y)^2}$$

So (*) becomes:

$$E[X] = p \sum_{i \geq i} iq^{i-1} = \frac{p}{(1 - q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

E.g.:

- $p=1/2$; on average head every 2\textsuperscript{nd} flip
- $p=1/10$; on average, head every 10\textsuperscript{th} flip.

How much would you pay to play?
Calculating $E[g(X)]$:

$Y = g(X)$ is a new r.v. Calc $P[Y=j]$, then apply defn:

$X = \text{sum of 2 dice rolls}$

$Y = g(X) = X \mod 5$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$p(i) = P[X=i]$</th>
<th>$i \cdot p(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$1/36$</td>
<td>$2/36$</td>
</tr>
<tr>
<td>3</td>
<td>$2/36$</td>
<td>$6/36$</td>
</tr>
<tr>
<td>4</td>
<td>$3/36$</td>
<td>$12/36$</td>
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<tr>
<td>5</td>
<td>$4/36$</td>
<td>$20/36$</td>
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<tr>
<td>6</td>
<td>$5/36$</td>
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<tr>
<td>7</td>
<td>$6/36$</td>
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<td>8</td>
<td>$5/36$</td>
<td>$40/36$</td>
</tr>
<tr>
<td>9</td>
<td>$4/36$</td>
<td>$36/36$</td>
</tr>
<tr>
<td>10</td>
<td>$3/36$</td>
<td>$30/36$</td>
</tr>
<tr>
<td>11</td>
<td>$2/36$</td>
<td>$22/36$</td>
</tr>
<tr>
<td>12</td>
<td>$1/36$</td>
<td>$12/36$</td>
</tr>
</tbody>
</table>

$E[X] = \sum_i ip(i) = 252/36 = 7$

$E[Y] = \sum_j jq(j) = 72/36 = 2$
expectation of a function of a random variable

Calculating $E[g(X)]$: Another way – add in a different order, using $P[X=\ldots]$ instead of calculating $P[Y=\ldots]$.

$X = \text{sum of 2 dice rolls}$

$Y = g(X) = X \mod 5$

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<td>6/36</td>
<td>12/36</td>
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<tr>
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<td>4/36</td>
<td>16/36</td>
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<tr>
<td>10</td>
<td>3/36</td>
<td>0/36</td>
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$E[Y] = \sum_j j \cdot q(j) = 72/36 = 2$

$E[g(X)] = \sum_i g(i) \cdot p(i) = 72/36 = 2$
Above example is not a fluke.

**Theorem:** if $Y = g(X)$, then $E[Y] = \sum_i g(x_i)p(x_i)$, where $x_i$, $i = 1, 2, ...$ are all possible values of $X$.

**Proof:** Let $y_j$, $j = 1, 2, ...$ be all possible values of $Y$.

\[
\sum g(x_i)p(x_i) = \sum_i \sum_{j: g(x_i)=y_j} g(x_i)p(x_i)
\]
\[
= \sum_j \sum_{i: g(x_i)=y_j} y_jp(x_i)
\]
\[
= \sum_j y_j \sum_{i: g(x_i)=y_j} p(x_i)
\]
\[
= \sum_j y_j P\{g(X) = y_j\}
\]
\[
= E[g(X)]
\]

Note that $S_i = \{ x_i \mid g(x_i)=y_i \}$ is a partition of the domain of $g$. 

**expectation of a function of a random variable**
A & B each bet $1, then flip 2 coins:

Let $X$ be A’s net gain: +1, 0, -1, resp.:

What is $E[X]$?

$$E[X] = 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{4} = 0$$

What is $E[X^2]$?

$$E[X^2] = 1^2 \cdot \frac{1}{4} + 0^2 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{4} = \frac{1}{2}$$
properties of expectation

Linearity of expectation, I

For any constants \( a, b \):

\[
E[aX + b] = aE[X] + b
\]

Proof:

\[
E[aX + b] = \sum_{x} (ax + b) \cdot p(x)
\]

\[
= a \sum_{x} xp(x) + b \sum_{x} p(x)
\]

\[
= aE[X] + b
\]

Example:

Q: In the 2-person coin game above, what is \( E[2X+1] \)?
A: \( E[2X+1] = 2E[X]+1 = 2\cdot0 + 1 = 1 \)
Let $X$ and $Y$ be two random variables derived from outcomes of a single experiment. Then

$$E[X+Y] = E[X] + E[Y]$$

True even if $X, Y$ dependent

**Proof:** Assume the sample space $S$ is countable. (The result is true without this assumption, but I won’t prove it.) Let $X(s), Y(s)$ be the values of these r.v.’s for outcome $s \in S$.

Claim: $E[X] = \sum_{s \in S} X(s) \cdot p(s)$

Proof: similar to that for “expectation of a function of an r.v.,” i.e., the events “$X=x$” partition $S$, so sum above can be rearranged to match the definition of $E[X] = \sum_x x \cdot P(X = x)$

Then:

$$E[X+Y] = \sum_{s \in S} (X[s] + Y[s]) \cdot p(s)$$

$$= \sum_{s \in S} X[s] \cdot p(s) + \sum_{s \in S} Y[s] \cdot p(s) = E[X] + E[Y]$$
Example

\(X\) = \# of heads in one coin flip, where \(P(X=1) = p\).

What is \(E(X)\)?

\[E[X] = 1 \cdot p + 0 \cdot (1-p) = p\]

Let \(X_i, 1 \leq i \leq n\), be \# of \(H\) in flip of coin with \(P(X_i=1) = p_i\)

What is the expected number of heads when all are flipped?

\[E[\Sigma_i X_i] = \Sigma_i E[X_i] = \Sigma_i p_i\]

Special case: \(p_1 = p_2 = ... = p\):

\[E[\# \text{ of heads in } n \text{ flips}] = pn\]
properties of expectation

Note:

Linearity is special!

It is *not* true in general that

\[ E[X \cdot Y] = E[X] \cdot E[Y] \]
\[ E[X^2] = E[X]^2 \]
\[ E[X/Y] = E[X] / E[Y] \]
\[ E[asinh(X)] = asinh(E[X]) \]