Exercises

1. Consider the process of rolling three six-sided dice. Represent the outcomes as $abc$ where each $a$, $b$, and $c$ can take values 1-6, where (die #1 = $a$), (die #2 = $b$), (die #3 = $c$)

(a) Determine the sample space $\Omega$ and calculate $|\Omega|$. For an event $abc$, what is $\Pr[abc]$?

(b) Consider the function $f : \Omega \to \mathbb{N}$, where for the event $x = abc \in \Omega$ we define $f$ by

$$f(x) = \text{the sum of the three dice} = a + b + c.$$ 

What is $f(y)$ for $y = 334 \leftrightarrow (\text{die #1} = 3), (\text{die #2} = 3), (\text{die #3} = 4)$? [super easy...]

(c) Consider the set $A = \{x \mid f(x) = 10\}$. Describe this set in words, and determine $|A|$ (the easiest way to do this is by brute force).

(d) Thinking abstractly, what if we wanted to associate probabilities with different values of $f$? How would you determine $\Pr[f(x) = 10]$? To interpret this, think about rolling the three dice, and then computing $f(x)$ based on the outcome. What is the probability of getting the value $f(x) = 10$?

2. To save space when computing the probabilities based on a function, let’s define an object called a random variable. Formally, a random variable $X$ is a function from the sample space to the real numbers. Instead of the typical $f(x)$ notation, since we only will care about the probabilities, we will write $X = x$ for the event that an outcome occurs which causes $X$ to take the value $x$.

(a) Explain how random variables relate to the definition of an event as a subset of the sample space.

(b) Consider the process of flipping a fair coin five times in a row. Let $Y$ be the random variable which maps a sequence of five coin flips to the number heads in the outcome. Calculate $\Pr[Y = k]$ for all $k = 0, 1, \ldots, 5$. What about for $k = 6$?

(c) We can also make sense of the event $Y \geq k$ in the natural way – it means the event that you flip the five coins and you get at least $k$ heads. Calculate $\Pr[Y \geq 2]$.

3. Define the expected value $E[X]$ (which is a real valued function of a random variable $X$) as

$$E[X] = \sum_{i=1}^{m} x_i \cdot \Pr[X = x_i].$$

where in this case $X$ only takes values in $\{x_1, \ldots, x_m\}$.

(a) Determine the meaning of the expression “real-valued function of a random variable.” In particular, why is it true that the expected value is a function of functions?

(b) Calculate $E[Y]$ for $Y$ described in (2b).

(c) Returning to (1), let $X$ be the random variable which equals the maximum value of the three dice. Calculate $\Pr[X = k]$ for $k = 1, 2, 3, 4, 5, 6$. Calculate $E[X]$. 
Challenge: Random Rooks.

1. Imagine that eight rooks are randomly placed on a standard $8 \times 8$ chessboard, where at most one rook can occupy a single space. Find the probability that all the rooks will be safe from one another, i.e. that there is no row or column with more than one rook.

2. As $n$ goes to infinity, what does the probability tend to? What does this mean?

3. Consider a three dimensional $8 \times 8 \times 8$ chessboard. Let’s label the 3D board by triples $(i, j, k)$ where each of $i, j, k$ range from 1 to 8. To generalize safety, we say that two rooks are safe from each other if there are no two dimensions in which both of these rooks’ positions agree. (in the 2D case, it was if any one dimension agreed).

If eight rooks are randomly placed on a $8 \times 8 \times 8$ chessboard, what’s the probability they will be safe from each other?

*Hint:* Last section we talked about a sequential method of calculating probabilities, where you imagine each part of the process happening in sequence, and multiply the probabilities at each step.

*Hint:* There’s a natural recursive structure to the problem. After placing one rook on an $8 \times 8$ chessboard...