CSE312, Spring 2011: Foundations of Computing II

Homework 3

Due in class on 13 May 2011

Guidelines: You can brainstorm with others, but please solve the problems and write up the answers by yourself. You may use textbooks, lecture notes, Wikipedia etc., but please don't use any other resources or references (e.g. online problem solutions) without asking.

1. In each of the following cases, are X and Y independent? Explain.

   (a)
   $$f(x,y) = \frac{xy^2}{30}, \qquad x = 1,2,3, \qquad y = 1,2.$$

   (b)
   $$f(x,y) = \frac{xy^2}{13}, \qquad (x,y) = (1,1),(1,2),(2,2).$$

   (c)
   $$f(x,y) = \frac{x^2}{28}, \qquad x = 1,2,3, \qquad y = 1,2.$$

   (d)
   $$f(x,y) = \frac{3}{2}x^2(1-|y|), \qquad -1 < x < 1, \qquad -1 < y < 1.$$

   [4+4+4+4]

2. Consider two random variables X and Y such that $\mathrm{Var}(X) + \mathrm{Var}(Y) = \mathrm{Var}(X+Y)$. Is it necessarily true that X and Y are independent? Why or why not?

   [4]

3. (a) You have a fair coin (probability of heads $= p_1 = 0.5$) and a highly biased coin (probability of heads $= p_2 = 0.05$). You flip each coin four times. Let X be the total number of heads. Find $\mathrm{E}[X]$ and $\mathrm{Var}(X)$.

   (b) You flip each coin 1000 times. (What is wrong with you?) Let Y be the total number of heads in the 2000 flips. Find $\mathrm{E}[Y]$ and $\mathrm{Var}(Y)$. Use a suitable approximation, and justify your choice.

   [5+5]

4. You have a biased coin, where the probability of heads is a random variable $X \sim \mathrm{Uni}(0,1)$.

   (a) You flip the coin 5 times, and see 2 heads and 3 tails. Given this new evidence, what are the variance and expectation of X?

   (b) You decide to keep flipping the coin until you see a head. This turns out to take seven flips. What are the updated variance and expectation of X? (Take the information from part (a) into account as well.)

   [5+5]

5. Now you have a fair six-sided die, and you roll it once. Let X be the indicator variable for an outcome of 1, and Y be the indicator for an outcome of 3.

   (a) Find the covariance and correlation between X and Y.

   (b) Find $\mathrm{Var}(X+Y)$.

   (c) What is the Covariance between X and Y given that the outcome is odd?

   (d) What is the Covariance between X and Y given that the outcome is even?

   [5+5+5+5]

6. Next, you roll your fair die 30 times. Let X be the number of ones and Y be the number of twos. Find the following:

   (a) The conditional p.m.f. of X, given $Y = y$.

   (b) $\mathrm{Var}(X|Y = y)$.

   (c) $\mathrm{Cov}(X, Y)$.

   (d) $E[X^2 - 4XY + Y^2]$.

   [5+5+5+5]

7. For $X \sim \mathrm{Exp}(2)$, upper bound $\mathrm{P}(X \geq 2)$ using the following. Express your answer as the ratio of the bound to the true value.

   (a) Markov's inequality

   (b) Chebyshev's inequality

   (c) The Chernoff bound

   [5+5+5]

8. Suppose you have 100 independent random variables $U_1, U_2, \ldots, U_{100}$ which are all uniformly distributed on $[0, 4]$. Define:

$$X = U_1 + U_2 + \cdots + U_{100}.$$

   Use the Central Limit Theorem to estimate the probability that X is in the range $[180, 230]$.

   [5]