3. Discrete Probability

CSE 312
Autumn 2011
W.L. Ruzzo
**Sample space:**  $S$ is the set of all possible outcomes of an experiment (Ω in your text book—Greek uppercase omega)

- **Coin flip:**  $S = \{\text{Heads, Tails}\}$
- **Flipping two coins:**  $S = \{(H,H), (H,T), (T,H), (T,T)\}$
- **Roll of one 6-sided die:**  $S = \{1, 2, 3, 4, 5, 6\}$
- **# emails in a day:**  $S = \{x : x \in \mathbb{Z}, \ x \geq 0 \}$
- **YouTube hrs. in a day:**  $S = \{x : x \in \mathbb{R}, 0 \leq x \leq 24 \}$
**Events:** $E \subseteq S$ is some subset of the sample space

- Coin flip is heads: $E = \{\text{Head}\}$
- At least one head in 2 flips: $E = \{(H,H), (H,T), (T,H)\}$
- Roll of die is 3 or less: $E = \{1, 2, 3\}$
- # emails in a day < 20: $E = \{x : x \in \mathbb{Z}, \; 0 \leq x < 20\}$
- Wasted day (>5 YT hrs): $E = \{x : x \in \mathbb{R}, \; x > 5\}$
set operations on events

E and F are events in the sample space $S$
set operations on events

E and F are events in the sample space S

Event “E OR F”, written $E \cup F$

$S = \{1, 2, 3, 4, 5, 6\}$
outcome of one die roll

$E = \{1, 2\}, \ F = \{2, 3\}$
$E \cup F = \{1, 2, 3\}$
set operations on events

E and F are events in the sample space $S$

Event “E AND F”, written $E \cap F$ or EF

$E = \{1,2\}$, $F = \{2,3\}$

$E \cap F = \{2\}$

$S = \{1,2,3,4,5,6\}$

outcome of one die roll
set operations on events

E and F are events in the sample space S

\[ EF = \emptyset \iff E, F \text{ are “mutually exclusive”} \]

\[ S = \{1,2,3,4,5,6\} \]

outcome of one die roll

\[ E = \{1,2\}, \; F = \{2,3\}, \; G = \{5,6\} \]

\[ EF = \{2\}, \text{ not mutually exclusive, but } E, G \text{ and } F, G \text{ are} \]
set operations on events

E and F are events in the sample space S

Event “not E,” written $\overline{E}$ or $\neg E$

$S = \{1, 2, 3, 4, 5, 6\}$
outcome of one die roll

$E = \{1, 2\}$  $\neg E = \{3, 4, 5, 6\}$
set operations on events

DeMorgan’s Laws

\[ \overline{E \cup F} = \overline{E} \cap \overline{F} \]

\[ \overline{E \cap F} = \overline{E} \cup \overline{F} \]
Intuition: Probability as the relative frequency of an event

$$\Pr(E) = \lim_{n \to \infty} \left( \frac{\text{# of occurrences of } E \text{ in } n \text{ trials}}{n} \right)$$

Axiom 1: $0 \leq \Pr(E) \leq 1$

Axiom 2: $\Pr(S) = 1$

Axiom 3: If $E$ and $F$ are mutually exclusive ($EF = \emptyset$), then

$$\Pr(E \cup F) = \Pr(E) + \Pr(F)$$

For any sequence $E_1, E_2, \ldots, E_n$ of mutually exclusive events,

$$\Pr \left( \bigcup_{i=1}^{n} E_i \right) = \Pr(E_1) + \cdots + \Pr(E_n)$$
implications of axioms

- \( \Pr(\overline{E}) = 1 - \Pr(E) \)

\[ \Pr(\overline{E}) = \Pr(S) - \Pr(E) \text{ because } S = E \cup \overline{E} \]

- If \( E \subseteq F \), then \( \Pr(E) \leq \Pr(F) \)

\[ \Pr(F') = \Pr(E) + \Pr(F - E) \geq \Pr(E') \]

- \( \Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(EF) \)

inclusion-exclusion formula

- And many others
equally likely outcomes

Simplest case: sample spaces with equally likely outcomes.

Coin flips: \( S = \{\text{Heads, Tails}\} \)

Flipping two coins: \( S = \{(H,H),(H,T),(T,H),(T,T)\} \)

Roll of 6-sided die: \( S = \{1, 2, 3, 4, 5, 6\} \)

Pr(each outcome) = \( \frac{1}{|S|} \)

In that case,

\[
\Pr(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|}
\]
Roll two 6-sided dice. What is \( \text{Pr}(\text{sum of dice} = 7) \) ?

\[ S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \} \]

\[ E = \{(6,1), (5,2), (4,3), (3,4), (2,5), (1,6)\} \]

\[ \text{Pr}(\text{sum} = 7) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}. \]
twinkies and ding dongs
4 Twinkies and 3 DingDongs in a bag. 3 drawn. What is \( \Pr(\text{one Twinkie and two DingDongs drawn}) \) ?

**Ordered:**

- Pick 3 ordered options: \( |S| = 7 \cdot 6 \cdot 5 = 210 \)
- Pick Twinkie as either 1\(^{\text{st}}\), 2\(^{\text{nd}}\), or 3\(^{\text{rd}}\) item:
  \[ |E| = (4 \cdot 3 \cdot 2) + (3 \cdot 4 \cdot 2) + (3 \cdot 2 \cdot 4) = 72 \]
- \( \Pr(\text{1 Twinkie and 2 DingDongs}) = \frac{72}{210} = \frac{12}{35} \).

**Unordered:**

- \( |S| = \binom{7}{3} = 35 \)
- \( |E| = \binom{4}{1} \binom{3}{2} = 12 \)
- \( \Pr(\text{1 Twinkie and 2 DingDongs}) = \frac{12}{35} \).
birthdays
What is the probability that, of $n$ people, none share the same birthday?

$|S| = (365)^n$

$|E| = (365)(364)(363)\cdots(365-n+1)$

$Pr(\text{no matching birthdays}) = \frac{|E|}{|S|} = \frac{(365)(364)\cdots(365-n+1)}{(365)^n}$

Some values of $n$…

$n = 23$: $Pr(\text{no matching birthdays}) < 0.5$

$n = 77$: $Pr(\text{no matching birthdays}) < 1/5000$

$n = 100$: $Pr(\text{no matching birthdays}) < 1/3,000,000$

$n = 150$: $Pr(\ldots) < 1/3,000,000,000,000,000,000$
n = 366?

Pr = 0

Above formula gives this, since

\[
\frac{(365)(364)\ldots(365-n+1)}{(365)^n} = 0
\]

when n = 366 (or greater).

Even easier to see via pigeon hole principle.
What is the probability that, of n people, none share the same birthday as you?

\[ |S| = (365)^n \]

\[ |E| = (364)^n \]

\[ \Pr(\text{no birthdays matches yours}) = \frac{|E|}{|S|} = \frac{(364)^n}{(365)^n} \]

Some values of n…

n = 23: \( \Pr(\text{no matching birthdays}) \approx 0.9388 \)

n = 77: \( \Pr(\text{no matching birthdays}) \approx 0.8096 \)

n = 253: \( \Pr(\text{no matching birthdays}) \approx 0.4995 \)
chip defect detection
n chips manufactured, one of which is defective
k chips randomly selected from n for testing

What is $\Pr(\text{defective chip is in k selected chips})$?

$$|S| = \binom{n}{k} \quad |E| = \binom{1}{1} \binom{n-1}{k-1}$$

$$\Pr(\text{defective chip is in k selected chips})$$

$$= \frac{\binom{1}{1} \binom{n-1}{k-1}}{\binom{n}{k}} = \frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{k!}{n!} = \frac{k}{n}$$
n chips manufactured, one of which is defective
k chips randomly selected from n for testing

What is \( \text{Pr(defective chip is in k selected chips)} \)?

Different analysis:

• Select k chips at random by permuting all n chips and then choosing the first k.
• Let \( E_i \) = event that \( i^{th} \) chip is defective.
• Events \( E_1, E_2, \ldots, E_k \) are mutually exclusive
• \( \text{Pr}(E_i) = 1/n \) for \( i=1,2,\ldots,k \)
• Thus \( \text{Pr(defective chip is selected)} \)
  \[ = \text{Pr}(E_1) + \cdots + \text{Pr}(E_k) = k/n. \]
n chips manufactured, two of which are defective. k chips randomly selected from n for testing.

What is Pr(a defective chip is in k selected chips) ?

\[ |S| = \binom{n}{k} \quad |E| = (1 \text{ chip defective}) + (2 \text{ chips defective}) \]

\[ = \binom{2}{1} \binom{n-2}{k-1} + \binom{2}{2} \binom{n-2}{k-2} \]

Pr(a defective chip is in k selected chips)

\[ = \frac{\binom{2}{1} \binom{n-2}{k-1} + \binom{2}{2} \binom{n-2}{k-2}}{\binom{n}{k}} \]
n chips manufactured, two of which are defective
k chips randomly selected from n for testing

What is $\text{Pr}(\text{a defective chip is in k selected chips})$?

Another approach:
$\text{Pr}(\text{a defective chip is in k selected chips}) = 1 - \text{Pr}(\text{none})$

$\text{Pr}(\text{none})$:

$$|S| = \binom{n}{k}, \ |E| = \binom{n-2}{k}, \ \text{Pr}(\text{none}) = \frac{\binom{n-2}{k}}{\binom{n}{k}}$$

$\text{Pr}(\text{a defective chip is in k selected chips}) = 1 - \frac{\binom{n-2}{k}}{\binom{n}{k}}$

(Same as above? Check it!)
poker hands
Consider 5 card poker hands.

A “straight” is 5 consecutive rank cards of any suit.

What is $Pr(\text{straight})$?

$$|S| = \binom{52}{5}$$

$$|E| = 10 \cdot \binom{4}{1}^5$$

$$Pr(\text{straight}) = \frac{10\binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394$$
52 card deck. Cards flipped one at a time.

After first ace (of any suit) appears, consider next card

\[ \Pr(\text{next card} = \text{ace of spades}) < \Pr(\text{next card} = 2 \text{ of clubs}) \]?

Maybe…

Case 1: Take Ace of Spades out of deck

Shuffle remaining 51 cards, add ace of spades after first ace

\(|S| = 52! \) (all cards shuffled)

\(|E| = 51! \) (only 1 place ace of spades can be added)

Case 2: Do the same thing with the 2 of clubs

\(|S| \) and \(|E| \) have same size

So,

\[ \Pr(\text{next} = \text{Ace of spades}) = \Pr(\text{next} = 2 \text{ of clubs}) = 1/52 \]
Ace of Spades: 2/6

2 of Clubs: 2/6

Theory is the same for a 3-card deck; \( \text{Pr} = \frac{2!}{3!} = \frac{1}{3} \)
hats
n persons at a party throw hats in middle, select at random. What is $\text{Pr}(\text{no one gets own hat})$?

$$\text{Pr}(\text{no one gets own hat}) = 1 - \text{Pr}(\text{someone gets own hat})$$

$$\text{Pr}(\text{someone gets own hat}) = \text{Pr}(\bigcup_{i=1}^{n} E_i), \text{ where } E_i = \text{ event that person } i \text{ gets own hat}$$

$$\text{Pr}(\bigcup_{i=1}^{n} E_i) = \sum_i \text{ P}(E_i) - \sum_{i<j} \text{ Pr}(E_i E_j) + \sum_{i<j<k} \text{ Pr}(E_i E_j E_k) \ldots$$
Visualizing the sample space $S$:

<table>
<thead>
<tr>
<th>People:</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hats:</td>
<td>$H_4$</td>
<td>$H_2$</td>
<td>$H_5$</td>
<td>$H_1$</td>
<td>$H_3$</td>
</tr>
</tbody>
</table>

I.e., a sample point is a permutation $\pi$ of $1, \ldots, n$

$$4 \ 2 \ 5 \ 1 \ 3$$

$|S| = n!$
E_i = event that person i gets own hat: \( \pi(i) = i \)

Counting single events:

|E_i| = (n-1)! for all i

Counting pairs:

\( E_i E_j : \pi(i) = i \) & \( \pi(j) = j \)

|E_i E_j| = (n-2)! for all i, j

A sample point in \( E_2 \) (also in \( E_5 \))

All points in \( E_2 \)

All points in \( E_2 \cap E_5 \)
n persons at a party throw hats in middle, select at random. What is \( \Pr(\text{no one gets own hat}) \)?

\( E_i = \) event that person \( i \) gets own hat

\[
\Pr(\bigcup_{i=1}^{n} E_i) = \sum_i \Pr(E_i) - \sum_{i<j} \Pr(E_i \cap E_j) + \sum_{i<j<k} \Pr(E_i \cap E_j \cap E_k) \ldots
\]

\( \Pr(k \text{ fixed people get own back}) = (n-k)!/n! \)

\[
\binom{n}{k} \times \text{that} = \frac{n!}{k!(n-k)!} \times \frac{(n-k)!}{n!} = \frac{1}{k!}
\]

\( \Pr(\text{none get own}) = 1 - \Pr(\text{some do}) = 1 - 1/1! + 1/2! - 1/3! + 1/4! \ldots + (-1)^n/n! \approx 1/e \approx .37 \)
Pr(none get own) = 1 - Pr(some do) = 
1 - 1 + 1/2! - 1/3! + 1/4! … + (-1)^n/n! ≈ e^{-1} ≈ .37

Oscillates forever, but quickly converges to 1/e