CSE 312 Foundations II

2. Counting

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How many ways are there to do X?

E.g., X = “choose an integer 1, 2, ..., 10”

E.g., X = “Walk from 1st & Marion to 5th & Pine, going only North or East at each intersection.”

The Point:

Counting gets hard when numbers are large, implicit and/or constraints are complex. Systematic approaches help.
If there are

\( n \) outcomes for some event A,

sequentially followed by \( m \) outcomes for event B,

then there are \( n \cdot m \) outcomes overall.

aka “The Product Rule”
Easily generalized to more events
Q. How many n-bit numbers are there?

A. \[2 \cdot 2 \cdot \ldots \cdot 2 = 2^n\]

Q. How many subsets of a set of size n are there?

A. 1^{st} \text{ member in or out}; 2^{nd} \text{ member in or out}, \ldots \Rightarrow 2^n
Q. How many 4-character passwords are there, if each character must be one of a, b, ..., z, 0, 1, ..., 9?

A. $36 \cdot 36 \cdot 36 \cdot 36 = 1,679,616 \approx 1.7$ million

Q. Ditto, but no character may be repeated?

A. $36 \cdot 35 \cdot 34 \cdot 33 = 1,413,720 \approx 1.4$ million

(And a non-mathematical question: why do security experts generally prefer schemes such as the second, even though it offers fewer choices?)
How many arrangements of 1, 2, 3 are possible (each used once, no repeat, order matters)?

More generally: How many arrangements of n distinct items are possible?

\[ n \cdot (n-1) \cdot (n-1) \cdot \ldots \cdot 1 = n! \quad (n \text{ factorial}) \]
Q. How many permutations of DOGIE are there?
A. $5! = 120$

Q. How many of DOGGY?
A. $5!/2! = 60$

Q. How many of GODOGGY?
A. \[
\frac{7!}{3!2!1!1!} = 420
\]
Q. Your elf-lord avatar can carry 3 objects chosen from
   1. sword
   2. knife
   3. staff
   4. water jug
   5. iPad w/magic WiFi

   How many ways can you equip him/her?

   A. \[
   \frac{5 \cdot 4 \cdot 3}{3!} = \frac{5!}{3! \cdot 2!} = 10
   \]
Combinations: number ways to choose \( r \) things from \( n \)

“n choose r” \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \) aka binomial coefficients

Important special case:
how many (unordered) pairs from \( n \) objects

\[
\binom{n}{2} = \frac{n(n-1)}{2} = \Theta(n^2)
\]

Many Identities. E.g.:

\[
\binom{n}{r} = \binom{n}{n-r} \quad \rightarrow \text{by symmetry of definition}
\]

\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \quad \rightarrow \text{1st object either in or out}
\]

\[
\binom{n}{r} = \frac{n}{r} \binom{n-1}{r-1} \quad \rightarrow \text{by definition + algebra}
\]
the binomial theorem

\[(x + y)^n = \sum_{k} \binom{n}{k} x^k y^{n-k}\]

proof 1: induction ...

proof 2: counting –

\[(x+y) \cdot (x+y) \cdot (x+y) \cdot \ldots \cdot (x+y)\]

pick either x or y from 1st binomial factor
pick either x or y from 2nd binomial factor

... 

pick either x or y from nth binomial factor

How many ways did you get exactly k x’s? \(\binom{n}{k}\)
another identity w/ binomial coefficients

\[ \sum_{k=0}^{n} \binom{n}{k} = 2^n \]

Proof:

\[ \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1 + 1)^n = 2^n \]
Q. How many ways are there to walk from 1st & Marion to 5th & Pine, going only North or East?

A: 7 choose 3 = 35:

*Changing the visualization often helps.* Instead of tracing paths on the grid above, list choices. You walk 7 blocks; at each intersection choose N or E; must choose N exactly 3 times.
\[ |A \cup B| = |A| + |B| - |A \cap B| \]
\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]

General: + singles - pairs + triples - quads + ...
If there are $n$ pigeons in $k$ holes and $n > k$, then some hole contains more than one pigeon. More precisely, some hole contains at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

There are two people in London who have the same number of hairs on their head.
Typical head $\sim 150,000$ hairs
Let’s say max-hairy-head $\sim 1,000,000$ hairs
Since there are more than 1,000,000 people in London…