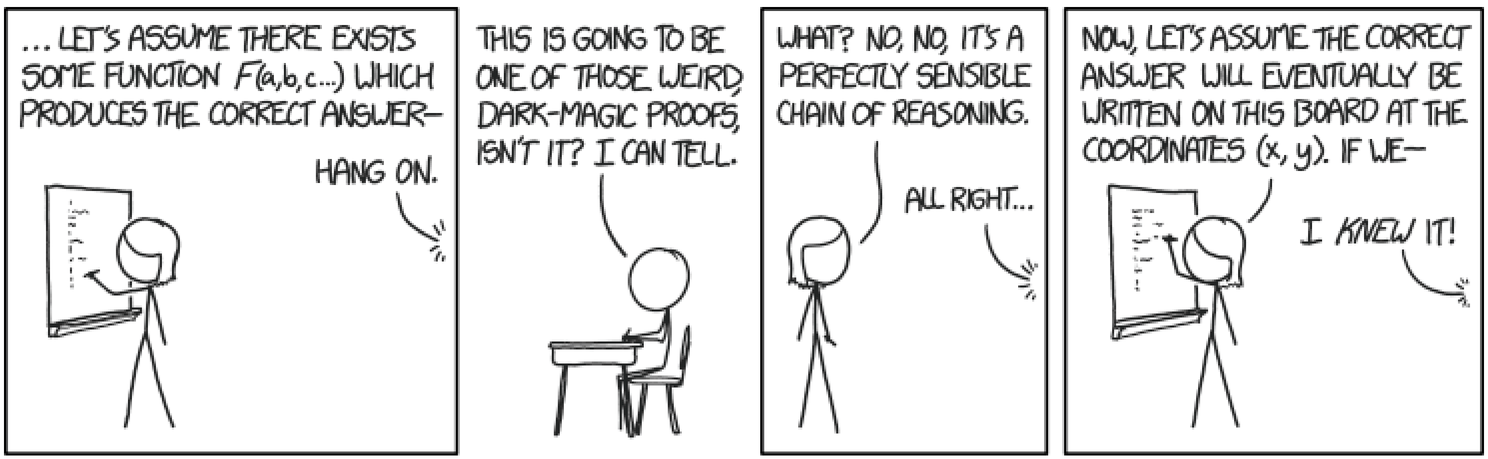
CSE 311: Foundations of Computing

Lecture 9: English Proofs, Strategies, Set Theory



Last class: Inference Rules for Quantifiers

Intro ∃

P(c) for some c

∴ ∃x P(x)

Elim ∀

∀x P(x)

∴ P(a) **for any** a

Intro ∀

"**Let** a **be arbitrary***"...P(a)

∴ ∀x P(x)

* in the domain of P. No other name in P depends on a

Elim ∃

∃x P(x)

∴ P(c) **for some *special********* c

** c is a NEW name.
List all dependencies for c.

Last class: Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove: "The square of every even number is even."

Formal proof of: $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

2.1	Even(**a**)	Assumption
2.2	$\exists y\ (\mathbf{a} = 2y)$	Definition of Even
2.3	$\mathbf{a} = 2\mathbf{b}$	Elim $\exists$: **b** special depends on **a**
2.4	$\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$	Algebra
2.5	$\exists y\ (\mathbf{a}^2 = 2y)$	Intro $\exists$ rule
2.6	Even($\mathbf{a}^2$)	Definition of Even

2. Even(**a**)$\rightarrow$Even($\mathbf{a}^2$) — Direct proof rule
3. $\forall x\ (\text{Even}(x)\rightarrow\text{Even}(x^2))$ — Intro $\forall$: 1,2

English Proof: Even and Odd

$\text{Even}(x) \equiv \exists y \; (x=2y)$
$\text{Odd}(x) \equiv \exists y \; (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

Proof: Let **a** be an arbitrary even integer.	1. Let **a** be an arbitrary integer 2.1 Even(**a**) Assumption
Then, by definition, **a** = 2**b** for some integer **b** (depending on **a**).	2.2 $\exists y\,(\mathbf{a} = 2y)$ Definition 2.3 **a** = 2**b** — **b** special depends on **a**
Squaring both sides, we get $\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$.	2.4 $\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$ Algebra
Since $2\mathbf{b}^2$ is an integer, by definition, $\mathbf{a}^2$ is even.	2.5 $\exists y\,(\mathbf{a}^2 = 2y)$ 2.6 Even($\mathbf{a}^2$) Definition
Since **a** was arbitrary, it follows that the square of every even number is even. ■	2. $\text{Even}(\mathbf{a}) \rightarrow \text{Even}(\mathbf{a}^2)$ 3. $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The square of every odd integer is odd."

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The square of every odd integer is odd."

Proof: Let b be an arbitrary odd integer.

Then, b = 2c+1 for some integer c (depending on b).

Therefore, $b^2 = (2c+1)^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1$.

Since $2c^2+2c$ is an integer, b^2 is odd. Since b was arbitrary, the square of every odd integer is odd. ■

Proof Strategies: Counterexamples

To *disprove* $\forall x\ P(x)$ prove $\exists \neg P(x)$:

- Works by de Morgan's Law: $\neg\forall x\ P(x) \equiv \exists x \neg P(x)$
- All we need to do that is find an x for which $P(x)$ is false
- This example is called a *counterexample* to $\forall x\ P(x)$.

e.g. Disprove "Every prime number is odd"

Proof Strategies: Proof by Contrapositive

If we assume $\neg q$ and derive $\neg p$, then we have proven $\neg q \rightarrow \neg p$, which is equivalent to proving $p \rightarrow q$.

	1.1. $\neg q$	Assumption
	...	
	1.3. $\neg p$	
1.	$\neg q \rightarrow \neg p$	Direct Proof Rule
2.	$p \rightarrow q$	Contrapositive: 1

Proof by Contradiction: One way to prove $\neg p$

If we assume p and derive F (a contradiction), then we have proven $\neg$p.

1.1. p Assumption

...

1.3. F

1. $p \rightarrow \mathbf{F}$ Direct Proof rule
2. $\neg p \vee \mathbf{F}$ Law of Implication: 1
3. $\neg p$ Identity: 2

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove: "No integer is both even and odd."

English proof: $\neg\, \exists x\ (\text{Even}(x) \wedge \text{Odd}(x))$

$\equiv \forall x\ \neg(\text{Even}(x) \wedge \text{Odd}(x))$

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove: "No integer is both even and odd."

English proof: $\neg \exists x\ (\text{Even}(x) \wedge \text{Odd}(x))$

$\equiv \forall x\ \neg(\text{Even}(x) \wedge \text{Odd}(x))$

Proof: We work by contradiction. Let x be an arbitrary integer and suppose that it is both even and odd. Then x=2a for some integer a and x=2b+1 for some integer b. Therefore 2a=2b+1 and hence a=b+½.

But two integers cannot differ by ½ so this is a contradiction. So, no integer is both even and odd. ∎

Rational Numbers

Domain of Discourse
Real Numbers

- **A real number x is *rational* iff there exist integers p and q with $q \neq 0$ such that $x = p/q$.**

$$\text{Rational}(x) \equiv \exists p\, \exists q\ \left((x=p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge q \neq 0\right)$$

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) $\equiv \exists p\ \exists q\ ((x = p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge (q \neq 0))$

Prove: "If x and y are rational then xy is rational."

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) $\equiv \exists p\ \exists q\ ((x = p/q) \wedge \mathrm{Integer}(p) \wedge \mathrm{Integer}(q) \wedge (q \neq 0))$

Prove: "If x and y are rational then xy is rational."

Proof: Let x and y be rational numbers. Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c,d, where d≠0.

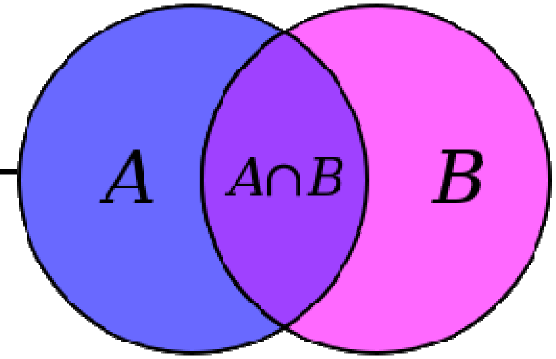
Multiplying, we get that xy = (ac)/(bd).

Since b and d are both non-zero, so is bd; furthermore, ac and bd are integers. It follows that xy is rational, by definition of rational. ■

Proofs

- Formal proofs follow simple well-defined rules and should be easy to check
  - In the same way that code should be easy to execute
- English proofs correspond to those rules but are designed to be easier for humans to read
  - Easily checkable in principle
- Simple proof strategies already do a lot
  - Later we will cover a specific strategy that applies to loops and recursion (mathematical induction)

Set Theory



Sets are collections of objects called elements.

Write $a \in B$ **to say that** a **is an element of set** B**,**
and $a \notin B$ **to say that it is not.**

Some simple examples
$A = \{1\}$
$B = \{1, 3, 2\}$
$C = \{\square, 1\}$
$D = \{\{17\}, 17\}$
$E = \{1, 2, 7, \text{cat}, \text{dog}, \varnothing, \alpha\}$

Some Common Sets

$\mathbb{N}$ is the set of **Natural Numbers;** $\mathbb{N} = \{0, 1, 2, \ldots\}$

$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48

$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, π, $\sqrt{2}$

[n] is the set **{1, 2, …, n}** when **n** is a natural number

{} = $\varnothing$ is the **empty set**; the *only* set with no elements

Sets can be elements of other sets

For example

$A = \{\{1\},\{2\},\{1,2\},\varnothing\}$

$B = \{1,2\}$

Then $B \in A$.

Definitions

- A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\, (x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall x\, (x \in A \rightarrow x \in B)$$

- Note: $(\boldsymbol{A} = \boldsymbol{B}) \equiv (\boldsymbol{A} \subseteq \boldsymbol{B}) \wedge (\boldsymbol{B} \subseteq \boldsymbol{A})$

Definition: Equality

A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\, (x \in A \leftrightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

Which sets are equal to each other?

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall\, x\, (x \in A \rightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}

<u>QUESTIONS</u>

$\varnothing \subseteq A$?
$A \subseteq B$?
$C \subseteq B$?

Building Sets from Predicates

S = the set of all* x for which P(x) is true

$$S = \{x : P(x)\}$$

S = the set of all x in A for which P(x) is true

$$S = \{x \in A : P(x)\}$$

*in the domain of P, usually called the "universe" U

Set Operations

$$A \cup B = \{ x : (x \in A) \vee (x \in B)\}$$ **Union**

$$A \cap B = \{ x : (x \in A) \wedge (x \in B)\}$$ **Intersection**

$$A \setminus B = \{ x : (x \in A) \wedge (x \notin B)\}$$ **Set Difference**

A = {1, 2, 3}
B = {3, 5, 6}
C = {3, 4}

QUESTIONS

Using A, B, C and set operations, make…

[6] =

{3} =

{1,2} =

More Set Operations

$$A \oplus B = \{\, x : (x \in A) \oplus (x \in B)\}$$

Symmetric Difference

$$\overline{A} = \{\, x : x \notin A \,\}$$

(with respect to universe U)

Complement

A = {1, 2, 3}
B = {1, 2, 4, 6}
Universe:
U = {1, 2, 3, 4, 5, 6}

$A \oplus B$ = {3, 4, 6}
$\overline{A}$ = {4,5,6}

It’s Boolean algebra again

- Definition for $\cup$ based on $\vee$

- Definition for $\cap$ based on $\wedge$

- Complement works like $\neg$

De Morgan's Laws

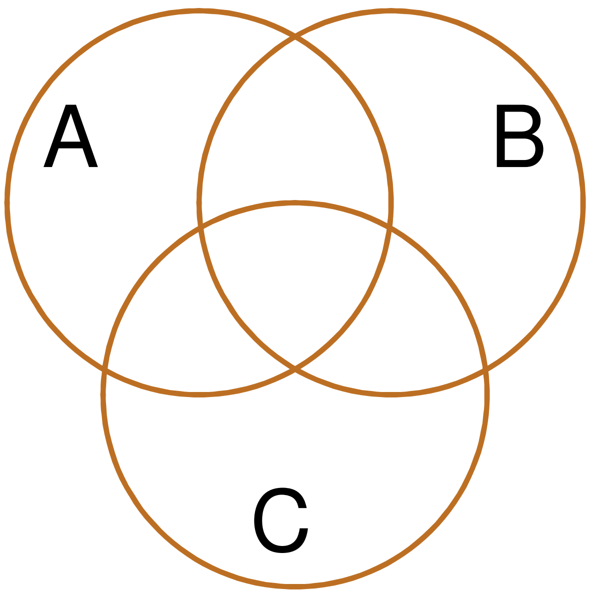
$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

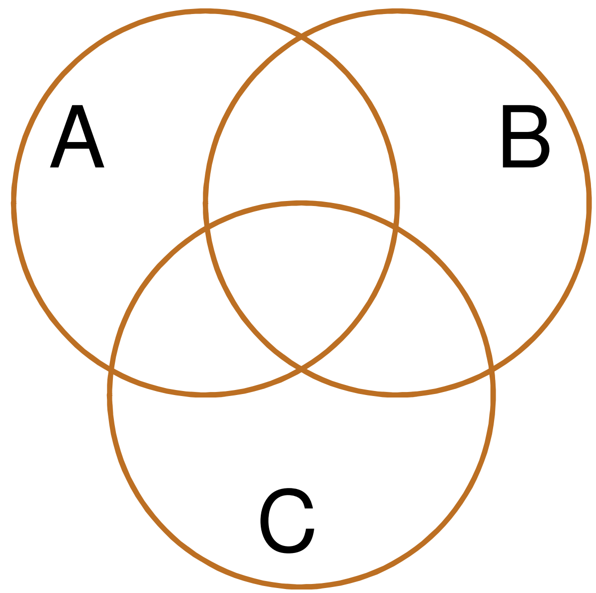
$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Proof technique:
To show C = D show
$x \in$ C $\rightarrow$ $x \in$ D and
$x \in$ D $\rightarrow$ $x \in$ C

Distributive Laws

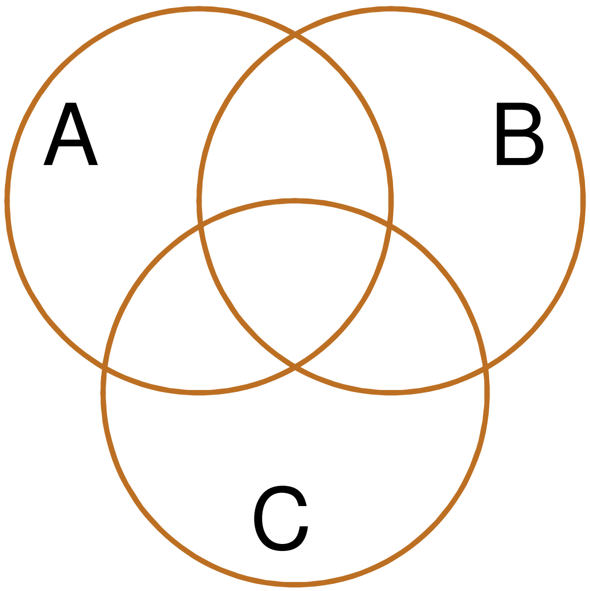
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

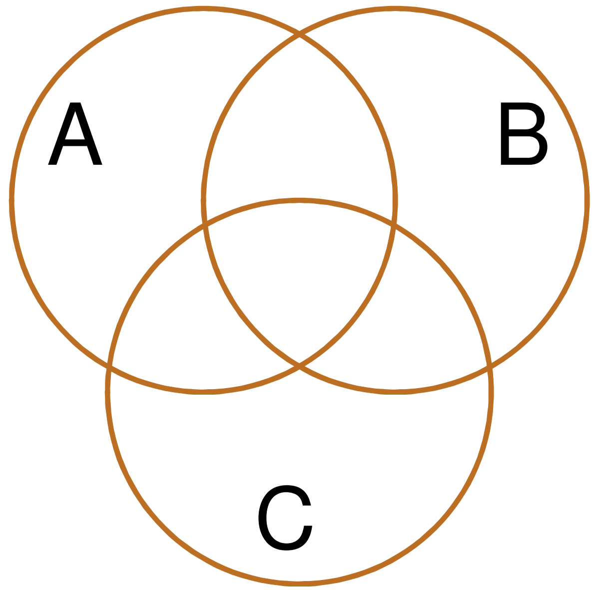




Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$





Power Set

- **Power Set of a set A = set of all subsets of A**

$$\mathcal{P}(A) = \{ B : B \subseteq A \}$$

- **e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class**

$\mathcal{P}$(Days)=?

$\mathcal{P}(\varnothing)$=?

Power Set

- **Power Set of a set A = set of all subsets of A**

$$\mathcal{P}(A) = \{ B : B \subseteq A \}$$

- **e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class**

$$\mathcal{P}(\text{Days})=\{\{M, W, F\}, \{M, W\}, \{M, F\}, \{W, F\}, \{M\}, \{W\}, \{F\}, \varnothing\}$$

$$\mathcal{P}(\varnothing)=\{\varnothing\} \neq \varnothing$$

Cartesian Product

$$A \times B = \{\, (a, b) \colon a \in A, b \in B \,\}$$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}.

$\boldsymbol{A} \times \emptyset = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A} \wedge \boldsymbol{b} \in \emptyset\} = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A} \wedge \mathbf{F}\} = \emptyset$

Representing Sets Using Bits

- Suppose universe U is $\{1,2, \dots, n\}$
- Can represent set $B \subseteq U$ as a vector of bits:

$$b_1 b_2 \dots b_n \text{ where } \quad \begin{array}{l} b_i = 1 \text{ when } i \in B \\ b_i = 0 \text{ when } i \notin B \end{array}$$

  – Called the *characteristic vector* of set B

- Given characteristic vectors for A and B
  – What is characteristic vector for $A \cup B$? $A \cap B$?

UNIX/Linux File Permissions

- `ls -l`

```
drwxr-xr-x ... Documents/
-rw-r--r-- ... file1
```

- Permissions maintained as bit vectors
  - Letter means bit is 1
  - “–” means bit is 0.

Bitwise Operations

$$\begin{array}{rl} & 01101101 \\ \vee & \underline{00110111} \\ & 01111111 \end{array}$$

Java: `z=x|y`

$$\begin{array}{rl} & 00101010 \\ \wedge & \underline{00001111} \\ & 00001010 \end{array}$$

Java: `z=x&y`

$$\begin{array}{rl} & 01101101 \\ \oplus & \underline{00110111} \\ & 01011010 \end{array}$$

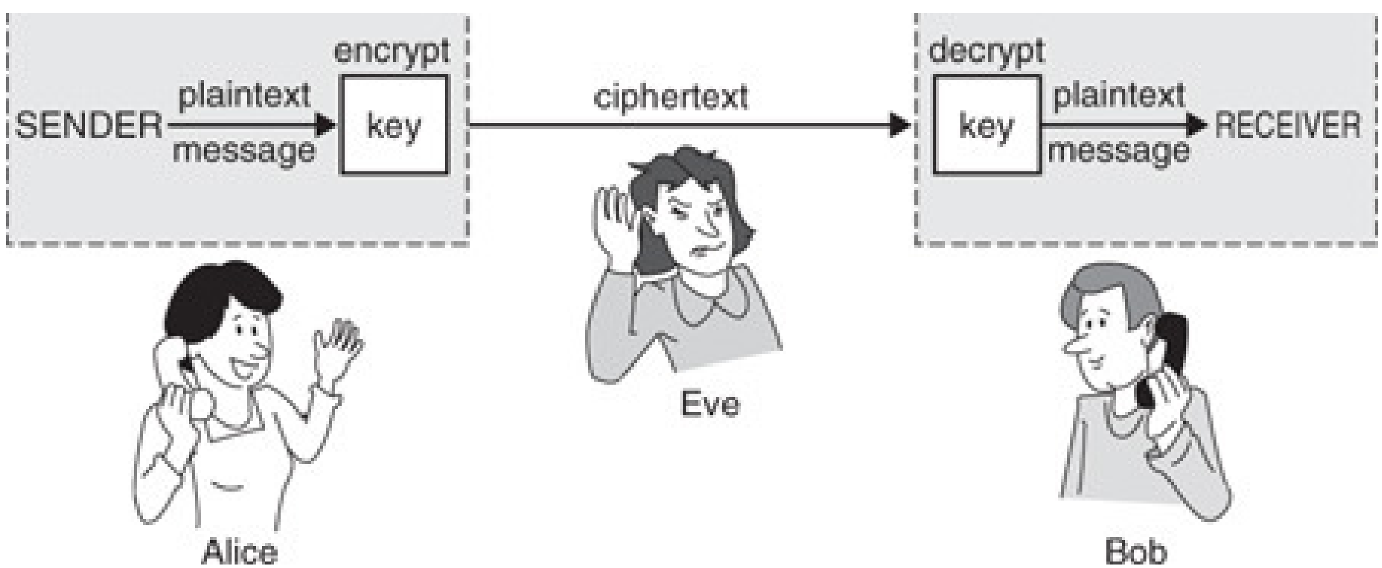
Java: `z=x^y`

A Useful Identity

- If x and y are bits: $(x \oplus y) \oplus y = ?$
- What if x and y are bit-vectors?

Private Key Cryptography

- Alice wants to communicate message secretly to Bob so that eavesdropper Eve who hears their conversation cannot tell what Alice's message is.
- Alice and Bob can get together and privately share a secret key K ahead of time.



One-Time Pad

- Alice and Bob privately share random n-bit vector K
  - Eve does not know K
- Later, Alice has n-bit message m to send to Bob
  - Alice computes $C = m \oplus K$
  - Alice sends C to Bob
  - Bob computes $m = C \oplus K$ which is $(m \oplus K) \oplus K$
- Eve cannot figure out m from C unless she can guess K

Russell's Paradox

$$S = \{ x : x \notin x \}$$

Suppose for contradiction that $S \in S$...

Russell's Paradox

$$S = \{ x : x \notin x \}$$

Suppose for contradiction that $S \in S$. Then, by definition of S, $S \notin S$, but that's a contradiction.

Suppose for contradiction that $S \notin S$. Then, by definition of the set S, $S \in S$, but that's a contradiction, too.

This is reminiscent of the truth value of the statement "This statement is false."