CSE 311: Foundations of Computing I

Practice Midterm Exam

Name:

ID #:

TA: **Section:**

INSTRUCTIONS:

- You have **50 minutes** to complete the exam.

- The exam is closed book. You may not use cell phones or calculators.

- All answers you want graded should be written on the exam paper.

- If you need extra space, use the back of a page. Make sure to mention that you did though.

- The problems are of varying difficulty.

- If you get stuck on a problem, move on and come back to it later.

Problem	Points	Score	Problem	Points	Score
1	20		5	5	
2	15		6	10	
3	10		7	20	
4	20				
			Σ	100	

Basic Techniques.

This part will test your ability to apply techniques that have been explicitly identified in lecture and reinforced through sections and homeworks. Remember to show your work and justify your claims.

1. **To Logic. . . or Not To Logic** [20 points]

(a) (5 points) Choose a meaning of $P(x, y, z)$ such that $\forall x \ \exists y \ \forall z \ P(x, y, z)$ is false, but $\forall x \ \forall y \ \exists z \ P(x, y, z)$ is true.

(b) (5 points) In the domain of integers, using any standard mathematical notation (but no new predicates), define $\text{Prime}(x)$ to mean "x is prime".

Let the predicates $D(x, y)$ mean "team x defeated team y" and $P(x, y)$ mean "team x has played team y." Give quantified formulas with the following meanings:

(c) (5 points) Every team has lost at least one game.

(d) (5 points) There is a team that has beaten every team it has played.

2. Obvious Induction Problem [15 points]

Prove for all $n \in \mathbb{N}$ that the following identity is true:

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

where $x \in \mathbb{R}, x \neq 1$.

3. 311 **is Prime!** [10 points]

Find all solutions in the range $0 \le x < 311$ to the modular equation:

$$12x \equiv 5 \pmod{311}$$

4. Even Circuits Are Fun [20 points]

The function multiple-of-three takes in two inputs: $(x_1x_0)_2$ and outputs 1 iff $3 \mid (x_1x_0)_2$.

(a) (5 points) Draw a table of values (e.g. a truth table) for multiple-of-three.

(b) (5 points) Write multiple-of-three as a sum-of-products.

(c) (5 points) Write multiple-of-three as a product-of-sums.

(d) (5 points) Write multiple-of-three as a simplified expression (don't bother explaining what rules you're using).

5. Irrationally Rational [5 points]

Recall the definition of irrational is that a number is not rational, and that

$$\text{Rational}(x) \equiv \exists p \; \exists q \; x = \frac{p}{q} \land \text{Integer}(p) \land \text{Integer}(q) \land q \neq 0$$

For this question, you may assume that π is irrational. Disprove that if x and y are irrational, then x + y is irrational.

6. Rationally Irrational [10 points]

Recall the definition of irrational is that a number is not rational, and that

$$\text{Rational}(x) \equiv \exists p \; \exists q \; x = \frac{p}{q} \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge q \neq 0$$

Prove that if x and y are rational and $x \neq 7$, then $\dfrac{y^2}{x - 7}$ is rational.

A Moment's Thought!

This section tests your ability to think a little bit more insightfully. The approaches necessary to solve these problems may not be immediately obvious. Remember to show your work and justify your claims.

7. Gotta □ m ∀ [20 points]

We say that k is *a square modulo* m iff there is some integer j such that $k \equiv j^2 \pmod{m}$.

Let $T = \{m \;:\; m = n^2 + 1 \text{ for some integer } n\}$.

(a) (8 points) Prove that if $m \in T$, then -1 is a square modulo m.

(b) (12 points) Prove that for all integers m and k, if $m \in T$ and k is a square modulo m then $-k$ is also a square modulo m.

CSE 311: Foundations of Computing I
Logical Equivalences Reference Sheet

Identity

$$p \wedge \mathsf{T} \equiv p$$
$$p \vee \mathsf{F} \equiv p$$

Domination

$$p \vee \mathsf{T} \equiv \mathsf{T}$$
$$p \wedge \mathsf{F} \equiv \mathsf{F}$$

Idempotency

$$p \vee p \equiv p$$
$$p \wedge p \equiv p$$

Commutativity

$$p \vee q \equiv q \vee p$$
$$p \wedge q \equiv q \wedge p$$

Associativity

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$
$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

Distributivity

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Absorption

$$p \vee (p \wedge q) \equiv p$$
$$p \wedge (p \vee q) \equiv p$$

Negation

$$p \vee \neg p \equiv \mathsf{T}$$
$$p \wedge \neg p \equiv \mathsf{F}$$

DeMorgan's Laws

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Double Negation

$$\neg\neg p \equiv p$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Axioms

Closure

$$a + b \text{ is in } \mathbb{B}$$
$$a \bullet b \text{ is in } \mathbb{B}$$

Commutativity

$$a + b = b + a$$
$$a \bullet b = b \bullet a$$

Associativity

$$a + (b + c) = (a + b) + c$$
$$a \bullet (b \bullet c) = (a \bullet b) \bullet c$$

Identity

$$a + 0 = a$$
$$a \bullet 1 = a$$

Distributivity

$$a + (b \bullet c) = (a + b) \bullet (a + c)$$
$$a \bullet (b + c) = (a \bullet b) + (a \bullet c)$$

Complementarity

$$a + a' = 1$$
$$a \bullet a' = 0$$

Theorems

Null

$$X + 1 = 1$$
$$X \bullet 0 = 0$$

Idempotency

$$X + X = X$$
$$X \bullet X = X$$

Involution

$$(X')' = X$$

Uniting

$$X \bullet Y + X \bullet Y' = X$$
$$(X + Y) \bullet (X + Y') = X$$

Absorbtion

$$X + X \bullet Y = X$$
$$(X + Y') \bullet Y = X \bullet Y$$
$$X \bullet (X + Y) = X$$
$$(X \bullet Y') + Y = X + Y$$

DeMorgan

$$(X + Y + \cdots)' = X' \bullet Y' \bullet \cdots$$
$$(X \bullet Y \bullet \cdots)' = X' + Y' + \cdots$$

Consensus

$$(X \bullet Y) + (Y \bullet Z) + (X' \bullet Z) = X \bullet Y + X' \bullet Z$$
$$(X + Y) \bullet (Y + Z) \bullet (X' + Z) = (X + Y) \bullet (X' + Z)$$

Factoring

$$(X + Y) \bullet (X' + Z) = X \bullet Z + X' \bullet Y$$
$$X \bullet Y + X' \bullet Z = (X + Z) \bullet (X' + Y)$$

Axioms & Inference Rules

Excluded Middle

$$\overline{}$$
$$\therefore \quad A \vee \neg A$$

Direct Proof

$$\frac{A \Rightarrow B}{\therefore \quad A \rightarrow B}$$

Modus Ponens

$$\frac{A \quad A \rightarrow B}{\therefore \qquad B}$$

Intro ∧

$$\frac{A \quad B}{\therefore \quad A \wedge B}$$

Elim ∧

$$\frac{A \wedge B}{\therefore \quad A \qquad B}$$

Intro ∨

$$\frac{A}{\therefore \ A \vee B \quad B \vee A}$$

Elim ∨

$$\frac{A \vee B \qquad \neg A}{\therefore \qquad B}$$

Intro ∀

$$\frac{\text{Let } a \text{ be an arbitrary} \dots}{\therefore \qquad \forall x \ P(x)}$$

Elim ∀

$$\frac{\forall x \ P(x)}{\therefore \ P(a) \text{ for any } a}$$

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \qquad \exists x \ P(x)}$$

Elim ∃

$$\frac{\exists x \ P(x)}{\therefore \ P(c) \text{ for some } special \ c}$$