CSE 311: Foundations of Computing I

Strong Induction Annotated Proofs

Relevant Definitions

Fibonacci Numbers — Definition

$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n > 1 \end{cases}$$

Bounding the Fibonacci Numbers

Prove that for all $n \in \mathbb{N} \setminus \{0, 1\}$, $2^{n/2-1} \leq f_n < 2^n$.

Proof	Commentary & Scratch Work
Let $P(n)$ be "$2^{n/2-1} \leq f_n < 2^n$" for all $n \in \mathbb{N} \setminus \{0, 1\}$. We go by strong induction on n.	*We're using strong induction because it's a recurrence.*
Base Case: Note that $2^{2/2-1} = 2^0 = 1 \leq 1 = 0 + 1 = f_0 + f_1 = f_2 < 4 = 2^2$. So, $P(2)$ is true.	
Induction Hypothesis: Suppose that $P(2) \wedge P(3) \wedge \cdots \wedge P(k)$ is true.	*An alternative, is to introduce a variable to range over the hypotheses. This would look like "Suppose that $P(\ell)$ is true for all $2 \leq \ell \leq k$ for some $k \in \mathbb{N} \setminus \{0, 1\}$."*
Induction Step: We show that $P(k + 1)$ is true.	*In strong induction, the IS takes careful planning. Whenever we attempt to use the IH, we need to make sure we've actually assumed it. In particular, we must ask:* • *What is the smallest value that k could be? (Here it's 2)* • *If it's the smallest value, can we plug into the recurrence for $k+1$? ($k + 1 = 2 + 1 = 3$, $f_3 = f_2 + f_1$)* • *If any of the values we'd need to plug in (here, 2 and 1) are less than our IH, (here, this is true for 1) we need special cases.*
Case ($(k + 1) - 2 < 2 \leftrightarrow k < 3 \leftrightarrow k = 2$): Note that $2^{3/2-1} = \frac{1}{2} \leq 2 = 1 + 1 = f_1 + f_2 = f_3 < 8 = 2^3$. So, $P(3)$ is true.	*Again, this case happened, because we can't apply the IH to f_1.*

Case ($(k+1)-2 \geq 2 \leftrightarrow k \geq 3$): Since $k \geq 3$, we know $f_{k+1} = f_k + f_{k-1}$. Furthermore, we know that $P(k)$ and $P(k-1)$ are both true by our IH. We take each piece of the claim independently. Note that $$\begin{aligned} f_{k+1} &= f_k + f_{k-1} \\ &\geq 2^{k/2-1} + 2^{(k-1)/2-1} \\ &\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1} \\ &= 2(2^{(k-1)/2-1}) \\ &= 2^{2/2+(k-1)/2-1} \\ &= 2^{(k+1)/2-1} \end{aligned}$$ Also, we have $$\begin{aligned} f_{k+1} &= f_k + f_{k-1} \\ &\leq 2^k + 2^{k-1} \\ &\leq 2^k + 2^k \\ &\leq 2^{k+1} \end{aligned}$$ Since the claim is true for both cases, $P(k) \to P(k+1)$. So, the claim is true for all $n \geq 2$ by induction on n.	*This is just a bunch of algebra. There's nothing special here other than the idea to just use the recurrence.*