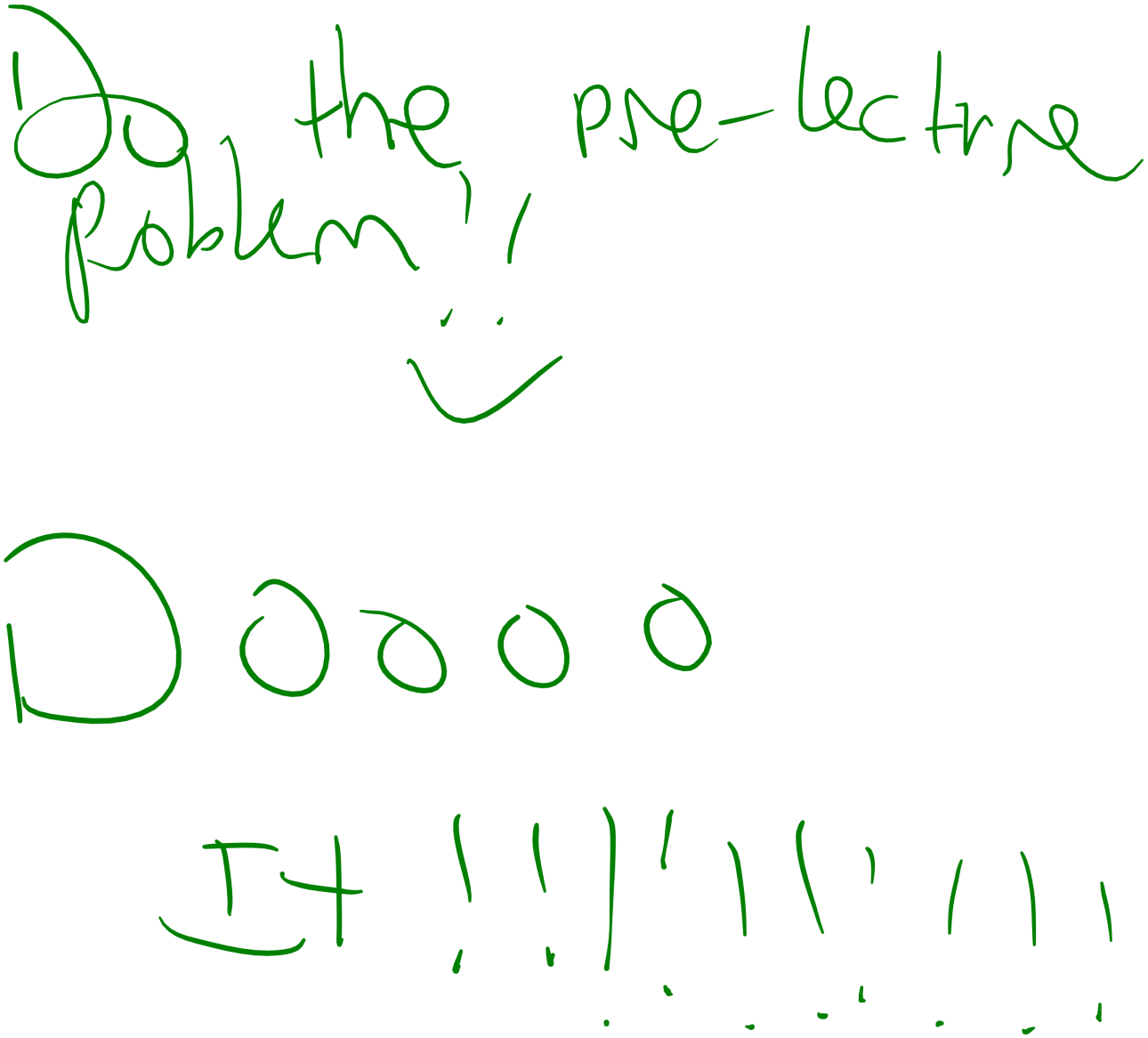
CSE 311

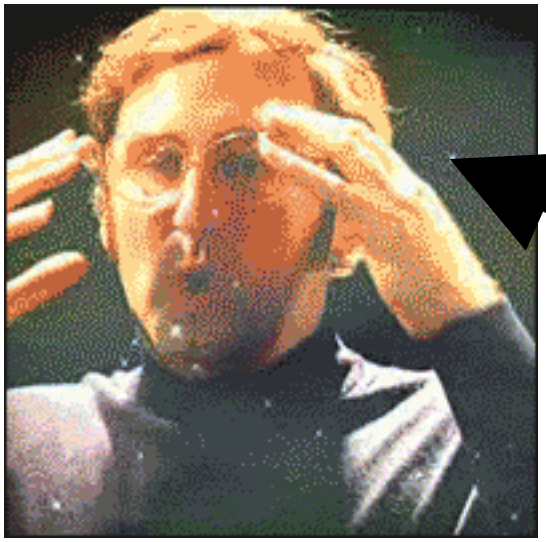
Foundations of Computing I

* All slides are a combined effort between previous instructors of the course

HW 3 De-Brief



HW 3 De-Brief



PROOFS!

HW 3 De-Brief

Proofs…

"This is hard"

"This is foreign"

"You didn't tell me exactly what to do…"

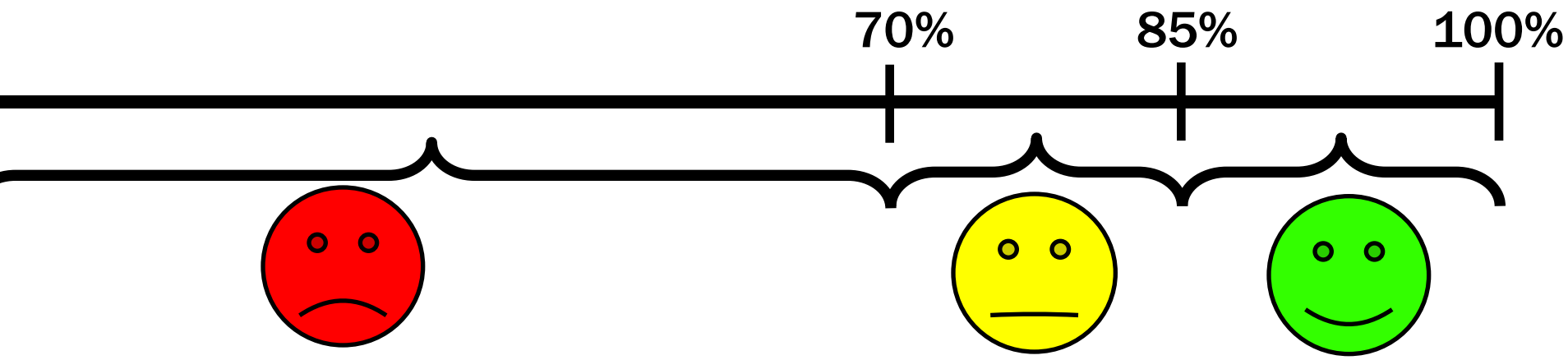
"I don't think I deserved to lose these points!"

HW 3 De-Brief

Think back to when you wrote your first essay.

HW 3 De-Brief

If your HW 3 score is in this range…



Schedule an appointment with Adam today. Not tomorrow; today.

There is something very important you didn't understand. If you aren't 100% on every comment you got by tomorrow, schedule an appointment with Adam.

Great Work! Read all your feedback.

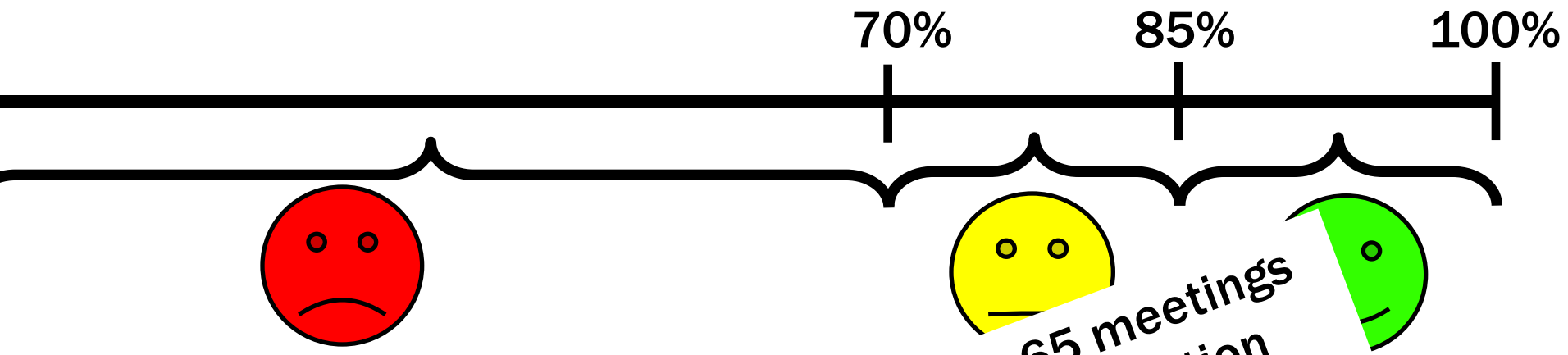
≈23%

≈33%

≈44%

HW 3 De-Brief

If your HW 3 score is in this range…



Schedule an appointment with Adam today. Not tomorrow; today.

I'm not joking. I expect to have about 65 meetings over the next week. There is a strong correlation between how students do on this HW, the next HW, and the midterm. Let's ensure you're not a statistic.

…ork!
…our
…eedback.

…ment
…u got by
tomorrow, schedule
an appointment
with Adam.

≈23%

≈33%

≈44%

HW 3 De-Brief

Okay, I got it. How do I schedule an appointment?

- Go to http://meeting.countablethoughts.com
- If I don't respond by Monday, then it probably didn't go through; so, e-mail me.

HW 3 De-Brief

“How I Oops 311”

- Never read the feedback, or

HW 3 De-Brief

“How I Oops 311”

- Never read the feedback, or
- Read the feedback but don’t take it seriously, or

HW 3 De-Brief

"How I Oops 311"

- Never read the feedback, or
- Read the feedback but don't take it seriously, or
- Read the feedback but convince yourself that "you get it now", or

HW 3 De-Brief

"How I Oops 311"

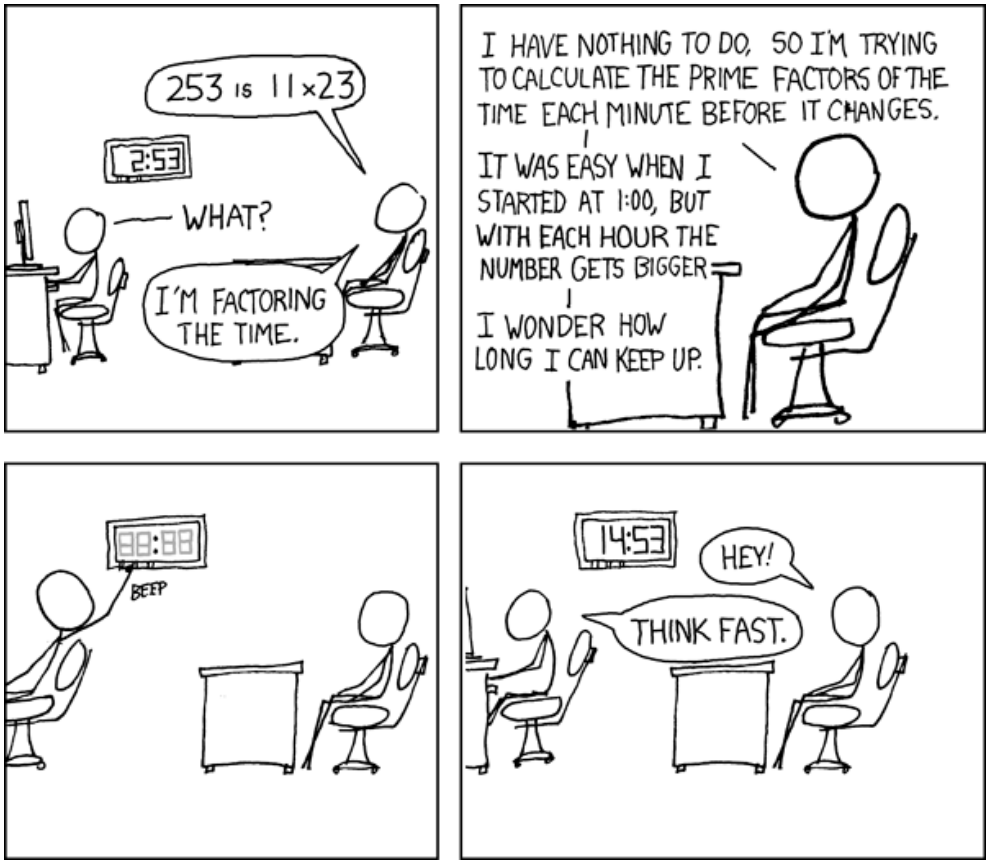
- Never read the feedback, or
- Read the feedback but don't take it seriously, or
- Read the feedback but convince yourself that "you get it now", or
- Read the feedback, talk to a TA, but don't apply what you've learned to future HWs, or…

HW 3 De-Brief

How smart you are and your grade are not the same thing.

CSE 311: Foundations of Computing

Lecture 12: Primes, GCD



Example

Let n be an integer.
Prove that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$

Case 1 (n is even):

Let's start by looking a a small example:

$0^2 = 0 \equiv 0 \pmod 4$
$1^2 = 1 \equiv 1 \pmod 4$
$2^2 = 4 \equiv 0 \pmod 4$
$3^2 = 9 \equiv 1 \pmod 4$
$4^2 = 16 \equiv 0 \pmod 4$

It looks like

$n \equiv 0 \pmod 2 \rightarrow n^2 \equiv 0 \pmod 4$, and
$n \equiv 1 \pmod 2 \rightarrow n^2 \equiv 1 \pmod 4$.

Case 2 (n is odd):

Suppose $n \equiv 1 \pmod 2$,

So, $2 \mid n-1$. So, $n-1 = 2k$

$n^2 = (2k+1)^2 = \ldots = 4\ell + 1$

Example

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$n \equiv 1 \pmod 2 \rightarrow n^2 \equiv 1 \pmod 4$.

Case 1 (n is even):

Suppose $n \equiv 0 \pmod 2$.
Then, $n = 2k$ for some k.
So, $n^2 = (2k)^2 = 4k^2$. So, by definition of congruence, $n^2 \equiv 0 \pmod 4$.

Case 2 (n is odd):

Suppose $n \equiv 1 \pmod 2$.
Then, $n = 2k + 1$ for some k.
So, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. So, by definition of congruence, $n^2 \equiv 1 \pmod 4$.

Primality

An integer p greater than 1 is called *prime* if the only positive factors of p are 1 and p.

A positive integer that is greater than 1 and is not prime is called *composite*.

Fundamental Theorem of Arithmetic

Every positive integer greater than 1 has a unique prime factorization

$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 2^4 \cdot 3$

$591 = 3 \cdot 197$

$45{,}523 = 45{,}523$

$321{,}950 = 2 \cdot 5 \cdot 5 \cdot 47 \cdot 137$

$1{,}234{,}567{,}890 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 3{,}607 \cdot 3{,}803$

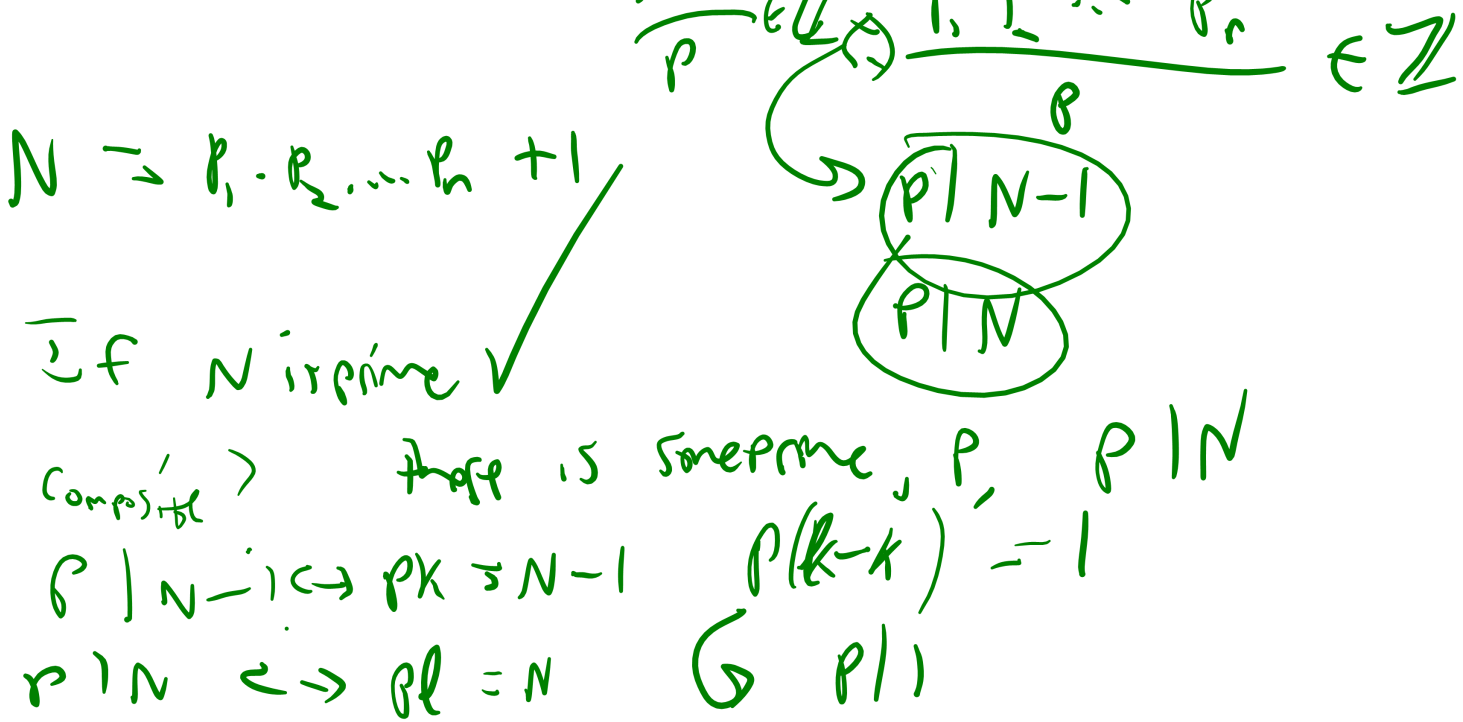
$n = p_0^{q_0} \cdot p_1^{q_1} \cdot \ldots \cdot p_m^{r_m}$

Euclid's Theorem

There are an infinite number of primes.

Proof by contradiction:

Suppose that there are only a finite number of primes: $p_1, p_2, \ldots, p_n$



Euclid's Theorem

There are an infinite number of primes.

Proof by contradiction:

Suppose for contradiction that there are n primes for some natural number n. Call them $p_1 < p_2 < \ldots < p_n$. Consider $P = p_1 p_2 \ldots p_n$, and define $Q = P + 1$.

Case 1 (Q is prime). Then, we're done, because Q is larger than any of the primes; so, it is a new prime.

Case 2 (Q is composite). Then, there must be some prime p such that $p \mid Q$. Note that since P divides every possible prime, $p \mid P$ as well. It follows that $p \mid (Q - P) \rightarrow p \mid ((P + 1) - P) \rightarrow p \mid 1$. This is impossible, because p must be at least two.

Since both cases lead to a contradiction, the original claim is true.

Famous Algorithmic Problems

- **Primality Testing**
  - Given an integer n, determine if n is prime
- **Factoring**
  - Given an integer n, determine the prime factorization of n

Factoring

Factor the following 232 digit number [RSA768]:

123018668453011775513049495838496272077
285356959533479219732245215172640050726
365751874520219978646938995647494277406
384592519255732630345373154826850791702
612214291346167042921431160222124047927
473779408066535141959745985690214341 3

123018668453011775513049495838496272077285356959533479219732245215172640050726365751874520219978646938995647494277406384592519255732630345373154826850791702612214291346167042921431160222124047927473779408066535141959745985690214341 3

=

33478071698956898786044169848212690817704794983713768568912431388982883793878002287614711652531743087737814467999489

×

3674604366679959042824463379962795263227915816434308764267603228381573966651127923337341714339681027009279873630891 7

Factoring

Okay. Fun...I guess...

It turns out this is hard.

Why should I care?

Factoring

We can use ("abuse") the fact that factoring is difficult for security.

Large Numbers

But...

```
Test.java:3: error: integer number too large:
12345678901234567890
        long x = 12345678901234567890;
                 ^
1 error
```

Thanks Java...

3674604366679959042824463379962795263227915816434308764267603228381573966651127923337341714339681027009279873630891 7

How do we store large numbers?

Large Non-negative Integer Representation

String

"12345"

$$\sum_{i=0}^{n} d_i 10^i$$

Sum

$5 \cdot 10^0 + 4 \cdot 10^1 + 3 \cdot 10^2 + 2 \cdot 10^3 + 1 \cdot 10^4$

List

```
[5, 4, 3, 2, 1]
```

Large Non-negative Integer Representation

String

“11000000111001”

$$\sum_{i=0}^{n} b_i 2^i$$

Sum

$$1 \cdot 2^0 + 0 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 + 1 \cdot 2^4 + 1 \cdot 2^5 + 0 \cdot 2^6 + 0 \cdot 2^7 + 0 \cdot 2^8 + 0 \cdot 2^9 + 0 \cdot 2^{10} + 0 \cdot 2^{11} + 1 \cdot 2^{12} + 1 \cdot 2^{13}$$

List

```
[1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1]
```

Large Non-negative Integer Operations MOD M

Just like before, we will consider these operations mod another number. This turns out to be "more general" and more useful.

So, instead of (a + b), we'll look at (a + b) mod m...

Large Non-negative Integer Operations MOD M

Addition

```
   [0, 0, 1, 1]
+  [1, 0, 0, 1]
   ---------------
   [1, 0, 1, 0, 1]
```

...and then we mod by m at the end.

Hey! This looks familiar!

Multiplication

This was HW, right?

...and then we mod by m at the end.

Large Non-negative Integer Operations MOD M

Subtraction?

Let's borrow from circuit design (again).

How does subtraction mod 2^32 work?

Two's Complement Representation

- For $0 < x \leq 2^{n-1}$, $-x$ is represented by the binary representation of $2^n - x$
- $\sim x = 2^n - 1 - x$
- $\sim x + 1 = 2^n - 1 - x + 1 = 2^n - x = -x$

```
  [0, 0, 1, 1]
- [1, 0, 0, 1]
  ----------------
  ??????????????
```

```
~ [1, 0, 0, 1]
  ----------------
  [0, 1, 1, 0]
+ [1]
  ----------------
  [1, 1, 1, 0]
```

Two's Complement Representation

```
   [0, 0, 1, 1]
-  [1, 0, 0, 1]
   ---------------
   ????????????????
```

$0 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 = 12$

$1 \cdot 2^0 + 0 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 = 9$

$8 \bmod 16 =$

$8 - 16$

-9

```
~ [1, 0, 0, 1]
  ------------
  [0, 1, 1, 0]
+ [1]
  ------------
  [1, 1, 1, 0]
```

```
   [0, 0, 1, 1]
+  [1, 1, 1, 0]
   ---------------
   [1, 1, 0, 0]
```

$0 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 = 12$

$1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3 = 7$

$1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 = 3$

Large Non-negative Integer Operations MOD M

Division?

This is weird. We'll come back to it next lecture.

Large Non-negative Integer Operations MOD M

Exponentiation?

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2						
3						
4						
5						
6						

$a^{16} = (a^2)^8$

a^2

$((a^2 \bmod m)^2 \bmod m)^2 \bmod m$

Exponentiation

- **Compute** 78365^{81453}

- **Compute** $78365^{81453} \bmod 104729$

- **Output is small**
  - need to keep intermediate results small

Repeated Squaring – small and fast

Since $a \bmod m \equiv a \pmod m$ for any a

we have $a^2 \bmod m = (a \bmod m)^2 \bmod m$

and $a^4 \bmod m = (a^2 \bmod m)^2 \bmod m$

and $a^8 \bmod m = (a^4 \bmod m)^2 \bmod m$

and $a^{16} \bmod m = (a^8 \bmod m)^2 \bmod m$

and $a^{32} \bmod m = (a^{16} \bmod m)^2 \bmod m$

Can compute $a^k \bmod m$ for $k=2^i$ in only i steps

Fast Exponentiation Algorithm

$$81453 = 2^{16} + 2^{13} + 2^{12} + 2^{11} + 2^{10} + 2^{9} + 2^{5} + 2^{3} + 2^{2} + 2^{0}$$

$$a^{81453} = a^{2^{16}} \cdot a^{2^{13}} \cdot a^{2^{12}} \cdot a^{2^{11}} \cdot a^{2^{10}} \cdot a^{2^{9}} \cdot a^{2^{5}} \cdot a^{2^{3}} \cdot a^{2^{2}} \cdot a^{2^{0}}$$

$a^{81453} \bmod m =$
$(\ldots(((((a^{2^{16}} \bmod m \cdot$
$a^{2^{13}} \bmod m) \bmod m \cdot$
$a^{2^{12}} \bmod m) \bmod m \cdot$
$a^{2^{11}} \bmod m) \bmod m \cdot$
$a^{2^{10}} \bmod m) \bmod m \cdot$
$a^{2^{9}} \bmod m) \bmod m \cdot$
$a^{2^{5}} \bmod m) \bmod m \cdot$
$a^{2^{3}} \bmod m) \bmod m \cdot$
$a^{2^{2}} \bmod m) \bmod m \cdot$
$a^{2^{0}} \bmod m) \bmod m$

The fast exponentiation algorithm computes $a^n \bmod m$ using $O(\log n)$ multiplications $\bmod m$

Fast Exponentiation

```
public static long FastModExp(long base, long exponent, long modulus) {
        long result = 1;
        base = base % modulus;

        while (exponent > 0) {
            if ((exponent % 2) == 1) {
                result = (result * base) % modulus;
                exponent -= 1;
            }
            /* Note that exponent is definitely divisible by 2 here. */
            exponent /= 2;
            base = (base * base) % modulus;
            /* The last iteration of the loop will always be exponent = 1 */
            /* so, result will always be correct. */
        }
        return result;
    }
```

P.S.: Pseudo-Random Number Generation

Linear Congruential method

$$x_{n+1} = (a\, x_n + c) \bmod m$$

Choose random x_0, a, c, m and produce a long sequence of x_n's

(Lagged Fibonacci Generator, Mersenne Twister...)