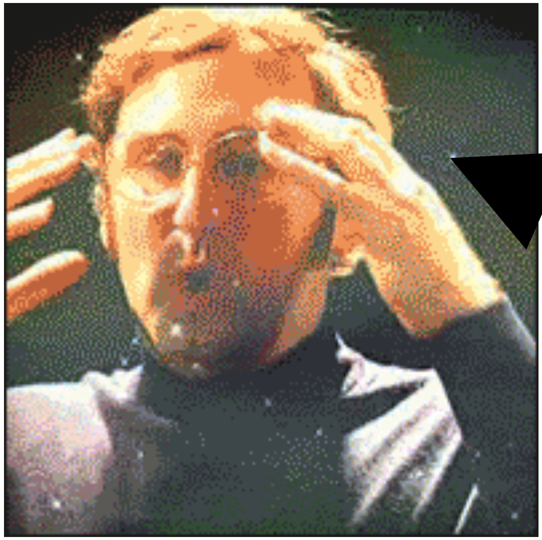
CSE 311

Foundations of Computing I

* All slides are a combined effort between previous instructors of the course

HW 3 De-Brief



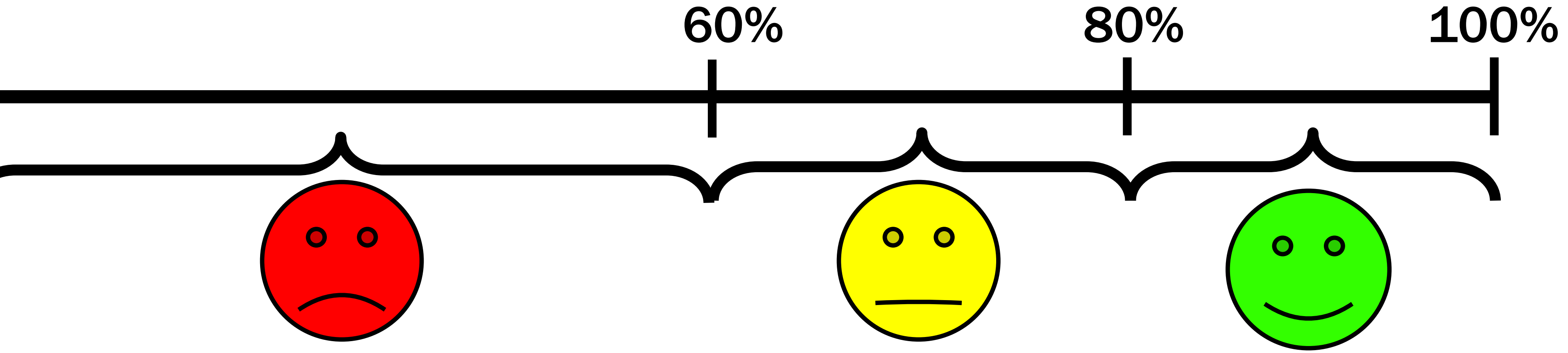
PROOFS!

HW 3 De-Brief

Think back to when you wrote your first essay.

HW 3 De-Brief

If your HW 3 score is in this range…



60% 80% 100%

Schedule an appointment with Adam today. Not tomorrow; today.

≈18%

There is something very important you didn’t understand. If you aren’t 100% on every comment you got by tomorrow, schedule an appointment with Adam.

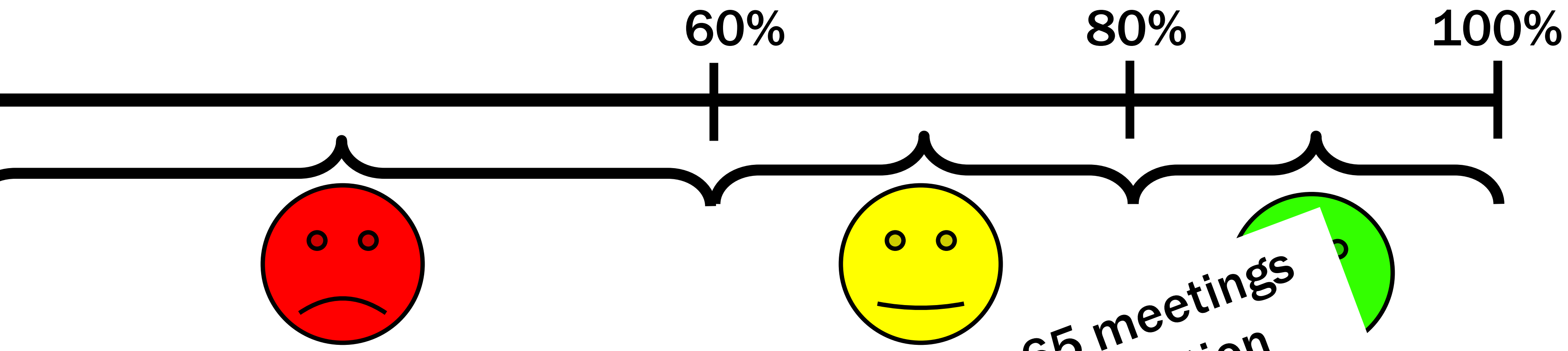
≈32%

Great Work! Read all your feedback.

≈50%

HW 3 De-Brief

If your HW 3 score is in this range...



Schedule an appointment with Adam today. Not tomorrow; today.

There

orrow,
chedule an
appointment with
Adam.

k!
ar
dback.

I'm not joking. I expect to have about 65 meetings over the next week. There is a strong correlation between how students do on this HW, the next HW, and the midterm. Let's ensure you're not a statistic.

≈18%

≈32%

≈50%

HW 3 De-Brief

Okay, I got it. How do I schedule an appointment?

- Go to http://meeting.countablethoughts.com
- If I don’t respond by Monday, then it probably didn’t go through; so, e-mail me.

HW 3 De-Brief

“How I Oops 311”

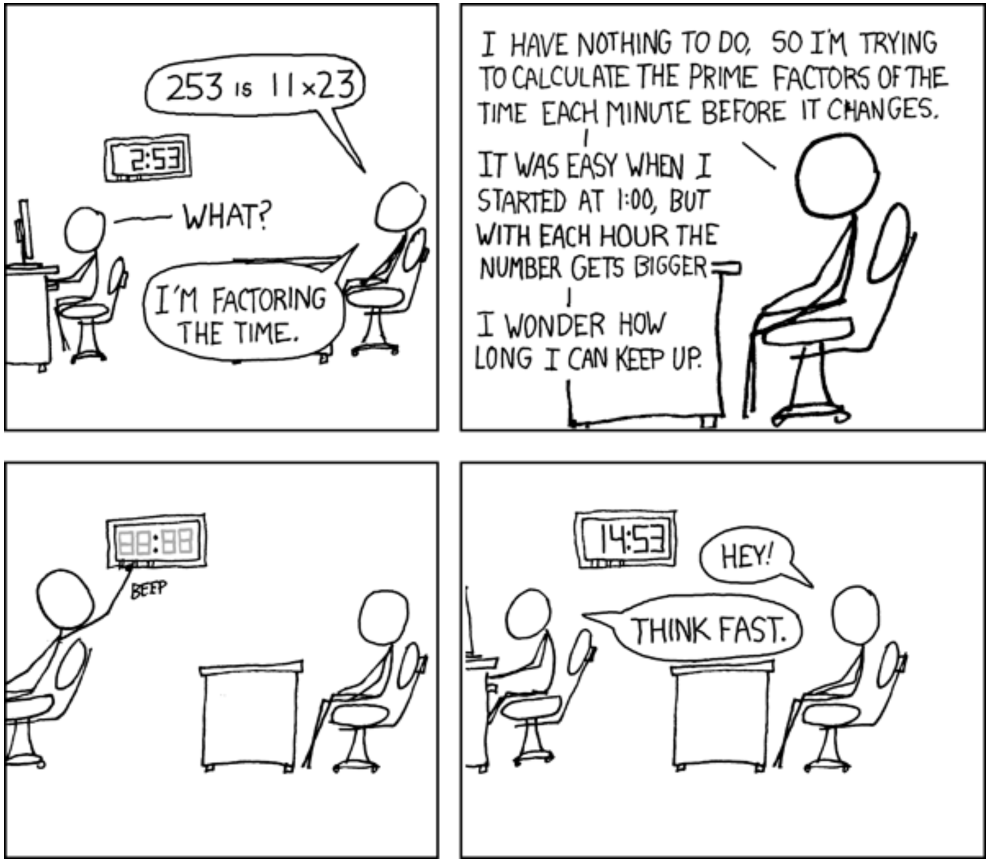
- Never read the feedback, or
- Read the feedback but don’t take it seriously, or
- Read the feedback but convince yourself that “you get it now”, or
- Read the feedback, talk to a TA, but don’t apply what you’ve learned to future HWs, or…

HW 3 De-Brief

How smart you are and your grade are not the same thing.

CSE 311: Foundations of Computing

Lecture 12: Primes, GCD


2:53
253 IS 11×23
WHAT?
I'M FACTORING THE TIME.
I HAVE NOTHING TO DO, SO I'M TRYING TO CALCULATE THE PRIME FACTORS OF THE TIME EACH MINUTE BEFORE IT CHANGES.
IT WAS EASY WHEN I STARTED AT 1:00, BUT WITH EACH HOUR THE NUMBER GETS BIGGER.
I WONDER HOW LONG I CAN KEEP UP.
88:88
BEEP
14:53
HEY!
THINK FAST.

Sign-Magnitude Integer Representation

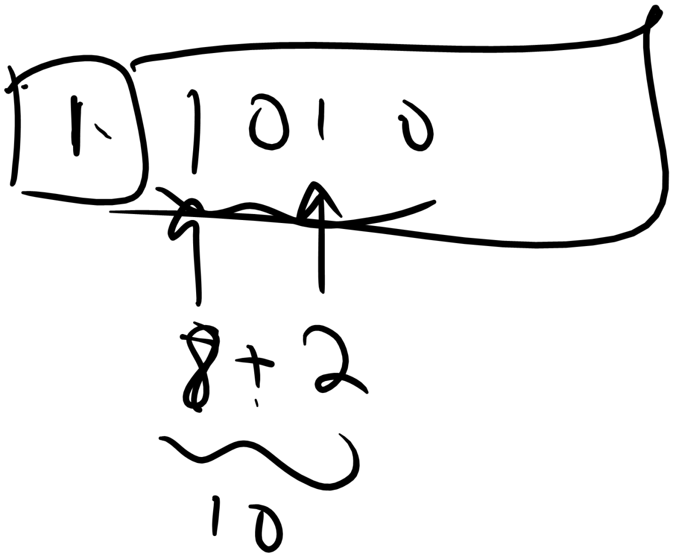
n-bit signed integers

Suppose $-2^{n-1} < x < 2^{n-1}$

First bit as the sign, n-1 bits for the value

99 = 64 + 32 + 2 + 1

18 = 16 + 2



For n = 8:

99: 0110 0011

-18: 1001 0010

Any problems with this representation?

Two’s Complement Representation

n bit signed integers, first bit will still be the sign bit

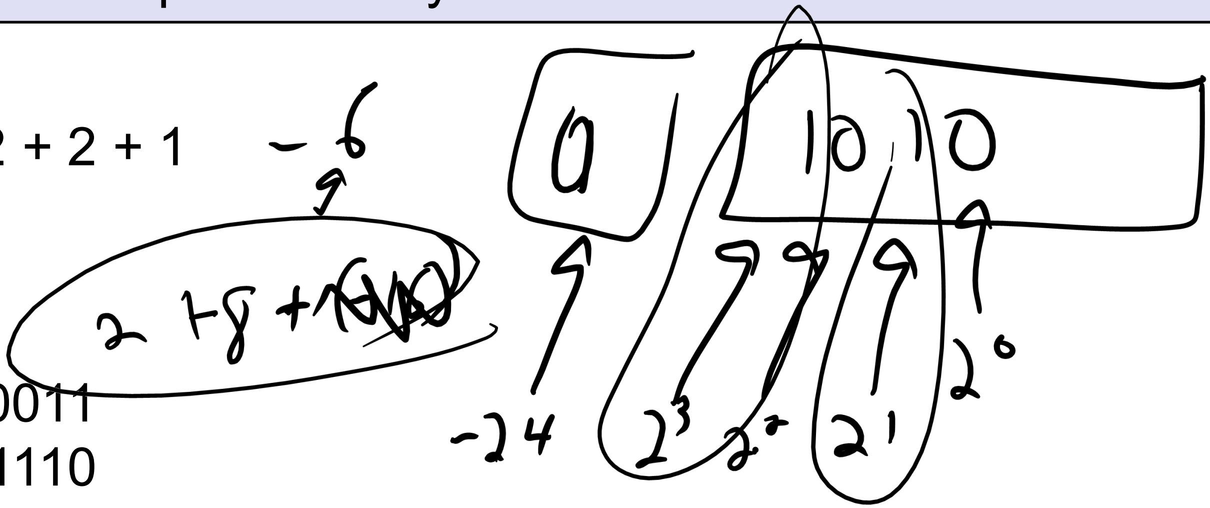
Suppose $0 \leq x < 2^{n-1}$,
x is represented by the binary representation of x

Suppose $0 \leq x \leq 2^{n-1}$,
$-x$ is represented by the binary representation of $2^n - x$

Key property: Twos complement representation of any number y is equivalent to y mod 2^n so arithmetic works mod 2^n

99 = 64 + 32 + 2 + 1
18 = 16 + 2

For n = 8:
99: 0110 0011
-18: 1110 1110



Sign-Magnitude vs. Two’s Complement

-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1111	1110	1101	1100	1011	1010	1001	0000	0001	0010	0011	0100	0101	0110	0111

Sign-bit

-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1000	1001	1010	1011	1100	1101	1110	1111	0000	0001	0010	0011	0100	0101	0110	0111

Two’s complement

Two’s Complement Representation

- For $0 < x \leq 2^{n-1}$, $-x$ is represented by the binary representation of $2^n - x$

- To compute this: Flip the bits of x then add 1:
  - All 1’s string is $2^n - 1$, so

    Flip the bits of x ≡ replace x by $2^n - 1 - x$

Basic Applications of mod

- Hashing
- Pseudo random number generation
- Simple cipher

Hashing

Scenario:

Map a small number of data values from a large domain $\{0, 1, \dots, M-1\}$ …

…into a small set of locations $\{0,1,\dots,n-1\}$ so one can quickly check if some value is present

- $\text{hash}(x) = x \bmod p$ for p a prime close to n
  - or $\text{hash}(x) = (ax+b) \bmod p$
- Depends on all of the bits of the data
  - helps avoid collisions due to similar values
  - need to manage them if they occur

Pseudo-Random Number Generation

Linear Congruential method

$$x_{n+1} = (a\, x_n + c) \bmod m$$

Choose random x_0, a, c, m and produce a long sequence of x_n's

Modular Exponentiation mod 7

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2						
3			6			
4						
5						
6						

$\boxed{a^b} \underline{\bmod m}$

$3^3 = 27 \bmod 7$

$= 6$

Exponentiation

- **Compute** 78365^{81453}

- **Compute** 78365^{81453} mod 104729

$((a \bmod m)(b \bmod m)) \bmod m = (ab) \bmod m$

$a \equiv b \pmod{m}$

$c \equiv d \pmod{m}$ → $ac \equiv bd \pmod{m}$

- **Output is small**
  - – need to keep intermediate results small

Repeated Squaring – small and fast

Since $a \bmod m \equiv a \pmod{m}$ for any a

$a^3 \bmod m$

we have $a^2 \bmod m = (a \bmod m)^2 \bmod m$ $((a)(a^2)) \bmod m$

and $a^4 \bmod m = (a^2 \bmod m)^2 \bmod m$

and $a^8 \bmod m = (a^4 \bmod m)^2 \bmod m$

and $a^{16} \bmod m = (a^8 \bmod m)^2 \bmod m$

and $a^{32} \bmod m = (a^{16} \bmod m)^2 \bmod m$

mod m

Can compute a^k mod m for $k=2^i$ in only i steps

$a^{32} = (a^{16})^2 = ((a^8)^2)^2 \bmod m$

Fast Exponentiation

$b^e \bmod m = (b^2)^{e/2} \bmod m$, when e is even)

$b^e \bmod m = (b*(b^{e-1} \bmod m) \bmod m)) \bmod m$

Fast Exponentiation

```
public static long FastModExp(long base, long exponent, long modulus) {
       long result = 1;
       base = base % modulus;

       while (exponent > 0) {
           if ((exponent % 2) == 1) {
               result = (result * base) % modulus;
               exponent -= 1;
           }
           /* Note that exponent is definitely divisible by 2 here. */
           exponent /= 2;
           base = (base * base) % modulus;
           /* The last iteration of the loop will always be exponent = 1 */
           /* so, result will always be correct. */
       }
       return result;
   }
```

$b^e \bmod m = (b^2)^{e/2} \bmod m$, when e is even)

$b^e \bmod m = (b*(b^{e-1} \bmod m) \bmod m)) \bmod m$

Program Trace

Let M = 104729

$$78365^{81453} \bmod M$$

$$= ((78365 \bmod M) * (78365^{81452} \bmod M)) \bmod M$$

$$= (78365 * ((78365^{2} \bmod M)^{81452/2} \bmod M)) \bmod M$$

$$= (78365 * ((78852)^{40726} \bmod M)) \bmod M$$

$$= (78365 * ((78852^{2} \bmod M)^{20363} \bmod M)) \bmod M$$

$$= (78365 * (86632^{20363} \bmod M)) \bmod M$$

$$= (78365 * ((86632 \bmod M) * (86632^{20362} \bmod M)) \bmod M$$

$$= \ldots$$

$$= 45235$$

Fast Exponentiation Algorithm

Another way:

$$81453 = 2^{16} + 2^{13} + 2^{12} + 2^{11} + 2^{10} + 2^{9} + 2^{5} + 2^{3} + 2^{2} + 2^{0}$$

$$a^{81453} = a^{2^{16}} \cdot a^{2^{13}} \cdot a^{2^{12}} \cdot a^{2^{11}} \cdot a^{2^{10}} \cdot a^{2^{9}} \cdot a^{2^{5}} \cdot a^{2^{3}} \cdot a^{2^{2}} \cdot a^{2^{0}}$$

$a^{81453} \bmod m =$

$(\ldots(((((a^{2^{16}} \bmod m \cdot$

$a^{2^{13}} \bmod m) \bmod m \cdot$

$a^{2^{12}} \bmod m) \bmod m \cdot$

$a^{2^{11}} \bmod m) \bmod m \cdot$

$a^{2^{10}} \bmod m) \bmod m \cdot$

$a^{2^{9}} \bmod m) \bmod m \cdot$

$a^{2^{5}} \bmod m) \bmod m \cdot$

$a^{2^{3}} \bmod m) \bmod m \cdot$

$a^{2^{2}} \bmod m) \bmod m \cdot$

$a^{2^{0}} \bmod m) \bmod m$

$(ab) \bmod m = ((a \bmod m)(b \bmod m)) \bmod m$

$(a^3) \bmod m$

The fast exponentiation algorithm computes $a^n \bmod m$ using $O(\log n)$ multiplications $\bmod m$

Primality

An integer *p* greater than 1 is called *prime* if the only positive factors of *p* are 1 and *p*.

311

A positive integer that is greater than 1 and is not prime is called *composite*.

$$\xcancel{\frac{311}{2}} \quad \xcancel{\frac{311}{3}} \quad \xcancel{\frac{311}{4}} \quad \cdots \quad \xcancel{\frac{311}{310}}$$

Fundamental Theorem of Arithmetic

Every positive integer greater than 1 has a unique prime factorization

$3 \cdot 5 = 15$

48 = 2 • 2 • 2 • 2 • 3

591 = 3 • 197

45,523 = 45,523

321,950 = 2 • 5 • 5 • 47 • 137

1,234,567,890 = 2 • 3 • 3 • 5 • 3,607 • 3,803

$$420 = 20 \cdot 21$$

$$= 2 \cdot 2 \cdot 5 \cdot 7 \cdot 3$$

$$= 2^2 \cdot 3 \cdot 5 \cdot 7$$

Euclid’s Theorem

There are an infinite number of primes.

2, 3 100, 101

527, 528

Proof by contradiction:

Suppose that there are only a finite number of primes: $p_1, p_2, \ldots, p_n$

$p_1 \cdot p_2 \cdots p_n = P$

$N = P + 1$

Case 1 (N prime):

$N > p_i$ ↯

q

$p_i \mid 1 \leftarrow (k-j)p_i = 1$

a , a+1

Case 2 (N composite):

$p_i \mid N \Rightarrow p_i \mid (P+1)$

$p_i \mid P$

$P+1 = kp_i$

$P = jp_i$

$\leftarrow jp_i = kp_i - 1$

Euclid’s Theorem

There are an infinite number of primes.

Proof by contradiction:

Suppose for contradiction that there are n primes for some natural number n. Call them $p_1 < p_2 < \ldots < p_n$. Consider $P = p_1 p_2 \ldots p_n$, and define $Q = P + 1$.

Case 1 (Q is prime). Then, we’re done, because Q is larger than any of the primes; so, it is a new prime.

Case 2 (Q is composite). Then, there must be some prime p such that $p \mid Q$. Note that since P divides every possible prime, $p \mid P$ as well. It follows that $p \mid (Q - P) \rightarrow p \mid ((P + 1) - P) \rightarrow p \mid 1$. This is impossible, because p must be at least two.

Since both cases lead to a contradiction, the original claim is true.

Famous Algorithmic Problems

- Primality Testing
  - Given an integer n, determine if n is prime
- Factoring
  - Given an integer n, determine the prime factorization of n

Factoring

Factor the following 232 digit number [RSA768]:

123018668453011775513049495838496272077
285356959533479219732245215172640050726
365751874520219978646938995647494277406
384592519255732630345373154826850791702
612214291346167042921431160222124047927
4737794080665351419597459856902143413

12301866845301177551304949583849627207728535695953347
92197322452151726400507263657518745202199786469389956
47494277406384592519255732630345373154826850791702612
21429134616704292143116022212404792747377940806653514
19597459856902143413

=

334780716989568987860441698482126908177047949837
137685689124313889828837938780022876147116525317
43087737814467999489

×

367460436667995904282446337996279526322791581643
430876426760322838157396665112792333734171433968
10270092798736308917

Factoring

Uh…fun?

Greatest Common Divisor

GCD(a, b):

Largest integer d such that $d \mid a$ and $d \mid b$

- GCD(100, 125) =
- GCD(17, 49) =
- GCD(11, 66) =
- GCD(13, 0) =
- GCD(180, 252) =

GCD and Factoring

$a = 2^3 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11 = 46{,}200$

$b = 2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 13 = 204{,}750$

$$\text{GCD}(a, b) = 2^{\min(3,1)} \cdot 3^{\min(1,2)} \cdot 5^{\min(2,3)} \cdot 7^{\min(1,1)} \cdot 11^{\min(1,0)} \cdot 13^{\min(0,1)}$$

Factoring is expensive!

Can we compute GCD(a,b) without factoring?