CSE 311

Foundations of Computing I

Pre-Lecture Problem

Translate the following into predicate logic:

"There is a student who is friends with every other student except her enemies."

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

"There is no largest integer"

$$\begin{aligned} \forall x \left(\neg(\forall y(x \geq y))\right) &\equiv \forall x \left(\exists y(\neg(x \geq y))\right) \\ &\equiv \forall x \left(\exists y\,(x < y)\right) \end{aligned}$$

"For every integer there is a larger integer"

Scope of Quantifiers

Example: NotLargest(x) $\equiv \exists y$ Greater (y, x)
$\equiv \exists z$ Greater (z, x)

Truth Value:

- **Doesn't depend on** y **or** z "**bound variables**"
- **Does depend on** x "**free variable**"

Quantifiers only act on free variables of the formulas they quantify, e.g. $\forall x(\exists y\ (P(x,y) \rightarrow \forall x\ Q(y, x)))$

Scope of Quantifiers

$\exists x\,(P(x) \land Q(x))$ **vs.** $\exists x\, P(x) \land \exists x\, Q(x)$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

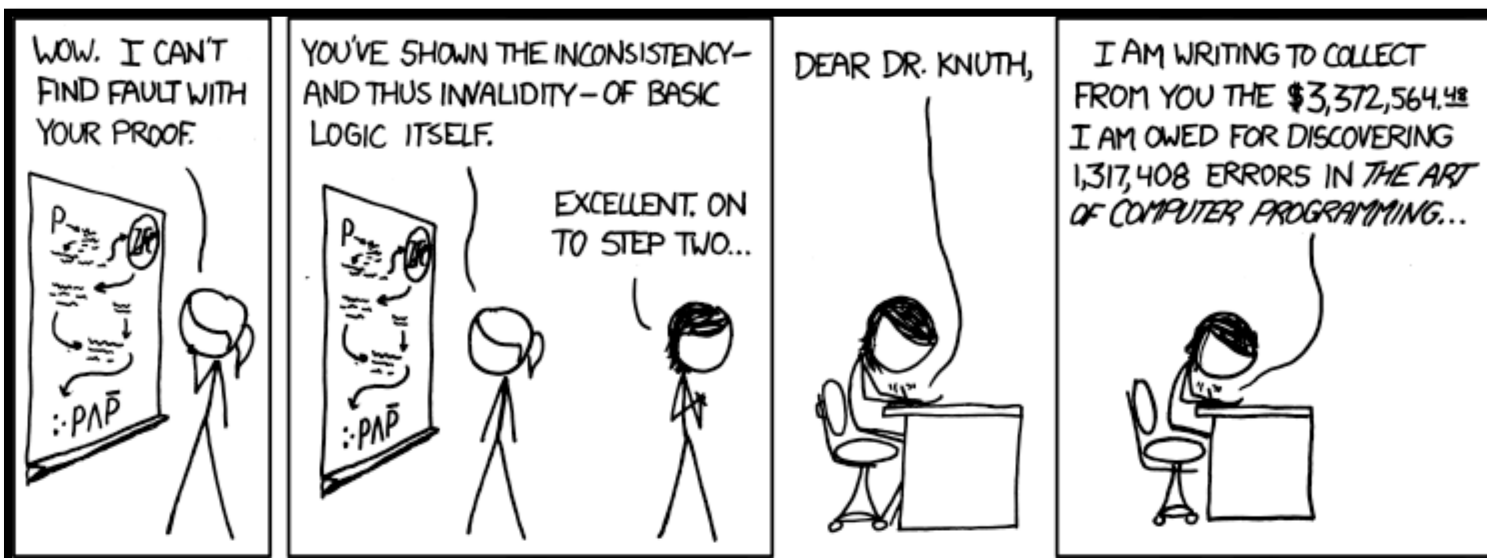
Quantifier "Style"

$$\forall x(\exists y\ (P(x,y) \rightarrow \forall x\ Q(y, x)))$$

This isn't "wrong", it's just horrible style.
Don't confuse your reader by using the same variable multiple times…there are a lot of letters…

CSE 311: Foundations of Computing

Lecture 6: Predicate Logic, Logical Inference



MMM Candy!

Let B be the "starred" bag of M&Ms.

Translate "There is a green M&M in B." into predicate logic.

Domain of Discourse
Colors & Bags

Predicate Definitions
Bag(x) ::= "x is a bag of M&Ms"
Color(x) ::= "x is a color"
Has(b, c) ::= "b has a c M&M."

$$\mathrm{Has}(\mathbf{Green}, \mathbf{B})$$

Notice that both "Green" and "B" are constants here! You could, instead, define a predicate Green(x):

$$\exists \mathrm{c}\big(\mathrm{Color}(c) \wedge \mathrm{Green}(c) \wedge \mathrm{Has}(c, \mathbf{B})\big)$$

MMM Candy!

Domain of Discourse
Colors & Bags

Predicate Definitions
Bag(x) ::= "x is a bag of M&Ms" Color(x) ::= "x is a color" Has(b, c) ::= "b has a c M&M."

Translate "Every bag of M&Ms has an M&M of some color."

$$\forall b \left(\mathrm{Bag}(b) \rightarrow \exists c\left(\mathrm{Color}(c) \land \mathrm{Has}(b, c)\right)\right)$$

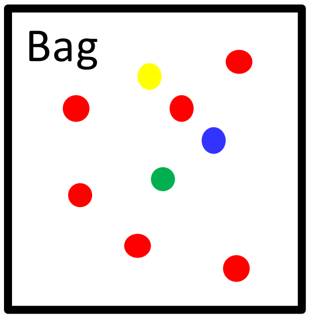
Translate "There is a color that all bags of M&Ms have."

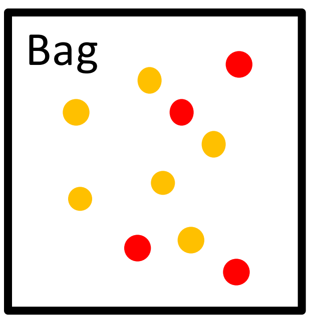
$$\exists c \left(\mathrm{Color}(c) \land \forall b\left(\mathrm{Bag}(b) \rightarrow \mathrm{Has}(b, c)\right)\right)$$

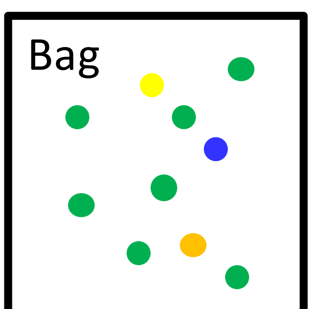
We're not done!

Domain of Discourse
Colors & Bags

Predicate Definitions
Bag(x) ::= "x is a bag of M&Ms"
Color(x) ::= "x is a color"
Has(b, c) ::= "b has a c M&M."







"There is a color that all bags of M&Ms have."

$$\exists c \left(\text{Color}(c) \land \forall b \big(\text{Bag}(b) \to \text{Has}(b, c) \big) \right)$$

For the bags on the left, this is not true. We need **a single color** that all the bags share.

vs.

"Every bag of M&Ms has an M&M of some color."

$$\forall b \left(\text{Bag}(b) \to \exists c \big(\text{Color}(c) \land \text{Has}(b, c) \big) \right)$$

For the bags on the left, this **is true**. The first bag has red; the second has orange, and the third has orange.

Still not done!

Domain of Discourse
Integers OR {1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= "x ≥ y"

"There is a number greater than or equal to all numbers."

$$\exists x \left(\forall y (\text{GreaterEq}(x, y)) \right)$$

"Every number has a number greater than or equal to it."

$$\forall y \left(\exists x (\text{GreaterEq}(x, y)) \right)$$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

The purple statement requires **an entire row** to be true.
The red statement requires one entry in **each column** to be true.

Nested Quantifiers

- **Bound variable names don't matter**

$$\forall x\ \exists y\ P(x, y) \equiv \forall a\ \exists b\ P(a, b)$$

- **Positions of quantifiers can sometimes change**

$$\forall x\ (Q(x) \wedge \exists y\ P(x, y)) \equiv \forall x\ \exists y\ (Q(x) \wedge P(x, y))$$

- **But: order is important...**

Quantification with Two Variables

expression	when **true**	when **false**
$\forall x \, \forall y \, P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \, \exists y \, P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \, \exists y \, P(x, y)$	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
$\exists y \, \forall x \, P(x, y)$	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.

Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other

- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

Why Proofs?

Consider $f(n) = 991n^2 + 1$

Is $f(n)$ a perfect square for any $n > 0$?

Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If p and p → q are both true then q must be true

- Write this rule as $$\frac{p,\ p \rightarrow q}{\therefore\ q}$$

- Given:
  - If it is Monday then you have a 311 class today.
  - It is Monday.

- Therefore, by modus ponens:
  - You have a 311 class today.

Proofs

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	given
2.	$p \rightarrow q$	given
3.	$q \rightarrow r$	given
4.	q	modus ponens from 1 and 2
5.	r	modus ponens from 3 and 4